On a particular class of self-decomposable random variables : the durations of Bessel excursions straddling independent exponential times.

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Abstract : The distributional properties of the duration of a recurrent Bessel process straddling an independent exponential time are studied in detail. Although our study may be considered as a particular case of M. Winkel's in [Wink], the infinite divisibility structure of these Bessel durations is particularly rich and we develop algebraic properties for a family of random variables arising from the Lévy measures of these durations.

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1 Introduction

1.1 The excursion durations $\Delta_\alpha (0<\alpha<1)$ of Bessel processes.

Let $\big((R_t, t\geq 0), P^{(\alpha)}\big)$ denote a Bessel process starting from 0, with dimension $d=2(1-\alpha)$, $0<d<2$ (or : $0<\alpha<1$). Let for any $t\geq 0$:

$$g_t^{(\alpha)} := \sup\{s\leq t\ ;\ R_s=0\} \qquad \text{and} \qquad d_t^{(\alpha)} := \inf\{s\geq t\ ;\ R_s=0\} \tag{1.1}$$

so that : $\Delta_t^{(\alpha)} := d_t^{(\alpha)} - g_t^{(\alpha)}$ is the length of the excursion above 0, straddling t, for the process $(R_u,\ u\geq 0)$.

We denote by $\mathfrak{e}$ a standard exponential variable, independent from $(R_u,\ u\geq 0)$. In a recent work, T. Fujita and M. Yor [F,Y] studied the laws of :

$$\sup_{s\leq g_{\mathfrak{e}}^{(\alpha)}} R_s, \quad \sup_{s\leq \mathfrak{e}} R_s, \quad \sup_{s\leq d_{\mathfrak{e}}^{(\alpha)}} R_s \tag{1.2}$$

Here, in a similar way, but focussing on durations, rather than on heights, we shall study exhaustively the law of :

$$\Delta_\alpha := \Delta_{\mathfrak{e}}^{(\alpha)} = d_{\mathfrak{e}}^{(\alpha)} - g_{\mathfrak{e}}^{(\alpha)} \tag{1.3}$$

In a first step we compute the density f_{Δ_α} of Δ_α :

$$f_{\Delta_\alpha}(x) = \frac{\alpha}{\Gamma(1-\alpha)}\, x^{-(\alpha+1)}(1-e^{-x})1_{(x\geq 0)} \tag{1.4}$$

and we prove that :

$$E\big[\exp(-\lambda\Delta_\alpha)\big] = (1+\lambda)^\alpha - \lambda^\alpha \qquad (\lambda\geq 0) \tag{1.5}$$

Note : We hope to devote another paper to the study of the remarkable properties of the subordinator $(\Delta_{1/2}(t),\ t\geq 0)$ whose value at time 1 is $\Delta_{1/2}$.

1.2 A general result by M. Winkel. ([Wink])

In fact, formulae (1.4) and (1.5) are a very particular case of a general result by M. Winkel [Wink], which we now describe.

Let $(\tau_l,\ l\geq 0)$ denote a subordinator with associated Bernstein function Φ, i.e.

$$E\big[\exp(-\lambda\tau_l)\big] = \exp\big(-l\Phi(\lambda)\big) \quad (\lambda, l\geq 0).$$

We define, for any $t\geq 0$:

$$L_t = \inf\{l : \tau_l > t\} \tag{1.6}$$

$$O_t = \tau_{(L_t)} - t \quad (\text{the overshoot})\ ;\ U_t = t - \tau_{(L_t)^-} \quad (\text{the undershoot}) \tag{1.7}$$

$$\text{and} \quad \Delta_t = \tau_{(L_t)} - \tau_{(L_t)^-} = O_t + U_t \tag{1.8}$$

For $\mathfrak{e}$, an independent standard exponential variable, M. Winkel computes the Laplace transform of the 7-tuple :

$$(\mathfrak{e},\ L_{\mathfrak{e}},\ U_{\mathfrak{e}},\ O_{\mathfrak{e}}, \tau_{L_{\mathfrak{e}}^-},\ \tau_{L_{\mathfrak{e}}},\ \Delta_{\mathfrak{e}}) \qquad \big(\text{see Corollary 1 in [Wink]}\big).$$

As a very partial result of this multidimensional formula, he obtains :

$$E\left[\exp(-\lambda\Delta_{\mathfrak{e}})\right] = \frac{\Phi(1+\lambda)-\Phi(\lambda)}{\Phi(1)} \qquad (\lambda \geq 0) \tag{1.9}$$

Hence, formula (1.5) is formula (1.9) applied to the subordinator $(\tau_l,\, l \geq 0)$ defined as :

$$\tau_l = \inf\{t \geq 0 \;:\; L_t > l\}$$

where (L_t) denotes the local time at 0 for the Bessel process $(R_t,\ t \geq 0)$ i.e. : $(\tau_l,\, l \geq 0)$ is a stable subordinator with index (α). We note that, from (1.9), we easily deduce the law of $\Delta_{\mathfrak{e}}$:

$$P(\Delta_{\mathfrak{e}} \in dx) = \frac{(1-e^{-x})}{\Phi(1)}\nu(dx) + \frac{c}{\Phi(1)}\delta_0(dx) \tag{1.10}$$

where ν denotes the Lévy measure of the subordinator $(\tau_l,\ l \geq 0)$ which admits c as its translation coefficient. There again, formula (1.4) is a particular case of (1.10) since the Lévy measure of the stable subordinator with index α is equal up to a multiplicative constant to : $(dx/x^{\alpha+1})1_{(x>0)}$.

To summarize : The formulae (1.4) and (1.5) are doubly particular cases of the results of M. Winkel, since :

- here, the subordinator $(\tau_l,\, l \geq 0)$ is a particular one, namely the α-stable subordinator;
- our formula only discusses the law of the r.v. $\Delta_{\mathfrak{e}}$, and not that of the 7-tuple :

$$(\mathfrak{e}, L_{\mathfrak{e}}, U_{\mathfrak{e}}, O_{\mathfrak{e}}, \tau_{(L_{\mathfrak{e}})^-}, \tau_{(L_{\mathfrak{e}})}, \Delta_{\mathfrak{e}})$$

1.3 The self-decomposability of the variable Δ_α $(0 < \alpha < 1)$.

Recall that a random variable Δ is said to be self-decomposable if, for any $c \in]0,1[$, there exists another variable $\Delta^{(c)}$ such that :

$$\Delta \overset{(\text{law})}{=} c\Delta + \Delta^{(c)} \tag{1.11}$$

where, on the RHS of (1.11), Δ and $\Delta^{(c)}$ are assumed independent. The class of self-decomposable laws (or variables) is a subclass of infinitely divisible laws; see, e.g., Sato [Sat].

In order to state our main result about the variable Δ_α, we need the following definition : let $\alpha > 0$, and K a positive r.v.. We shall say that $(Y_t,\ t \geq 0)$ is an (α, K) compound Poisson process (valued in $\mathbb{R}_+$) if :

$$Y_t := \sum_{i=1}^{N_t} K_i \tag{1.12}$$

where $(K_1, K_2, \cdots)$ is a sequence of i.i.d. variables, distributed as K, and with $(N_t,\ t \geq 0)$ a Poisson process with parameter α independent of the sequence $(K_i,\ i = 1, 2, \cdots)$. In particular, N_t is a Poisson variable with parameter (αt).

Theorem 1.1
For any $\alpha \in]0,1[$, one has :

$$1)\quad i)\quad \Delta_\alpha \overset{\text{(law)}}{=} \frac{\gamma_{(1-\alpha)}}{\beta_{\alpha,1}} \overset{\text{(law)}}{=} \frac{\gamma_{(1-\alpha)}}{U^{1/\alpha}} \tag{1.13}$$

where, on the RHS of (1.13), $\gamma_{(1-\alpha)}$ and $\beta_{\alpha,1}$ are two independent r.v's with respective laws gamma $(1-\alpha)$ and beta $(\alpha,1)$, and U denotes a uniform variable on [0,1], independent from $\gamma_{(1-\alpha)}$.

ii) The density of Δ_α, denoted here by f_{Δ_α}, is given by :

$$f_{\Delta_\alpha}(x) = \frac{\alpha}{\Gamma(1-\alpha)}\, x^{-\alpha-1}(1-e^{-x})1_{x\geq 0} \tag{1.14}$$

iii) The Laplace transform of (the law of) Δ_α is :

$$E(e^{-\lambda\Delta_\alpha}) = (1+\lambda)^\alpha - \lambda^\alpha \quad (\lambda \geq 0) \tag{1.15}$$

2) i) Δ_α is self decomposable, and the Lévy-Khintchine formula writes :

$$E(e^{-\lambda\Delta_\alpha}) = \exp\big(-(1-\alpha)\int_0^\infty (1-e^{-\lambda x})\, E(e^{-xG_\alpha})\frac{dx}{x}\big) \tag{1.16}$$

where, in (1.16), G_α denotes a r.v. with values in $[0,1]$, and density :

$$f_{G_\alpha}(u) = \frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi}\, \frac{u^{\alpha-1}(1-u)^{\alpha-1}}{(1-u)^{2\alpha} - 2(1-u)^\alpha u^\alpha\cos(\pi\alpha) + u^{2\alpha}}\, 1_{[0,1]}(u) \tag{1.17}$$

ii) The r.v. G_α is characterized by its Stieltjes transform :

$$\begin{aligned} S(f_{G_\alpha})(\lambda) &:= \int_0^\infty \frac{f_{G_\alpha}(u)}{\lambda+u}du = E\Big(\frac{1}{\lambda+G_\alpha}\Big) \\ &= \frac{\alpha}{1-\alpha}\,\frac{\lambda^{\alpha-1}-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-\lambda^\alpha} \quad (\lambda\geq 0) \end{aligned} \tag{1.18}$$

or, equivalently by :

$$E[e^{-\lambda\mathfrak{e}G_\alpha}] = E\Big(\frac{1}{1+\lambda G_\alpha}\Big) = \frac{\alpha}{1-\alpha}\,\frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha - 1} \quad (\lambda\geq 0) \tag{1.19}$$

3) Define (the law of) the r.v. :

$$K_\alpha \overset{\text{(law)}}{=} \mathfrak{e}/G_\alpha \tag{1.20}$$

where, on the RHS of (1.20) $\mathfrak{e}$ and G_α are assumed independent. In particular :

$$P(K_\alpha \geq\ x) = P\Big(\frac{\mathfrak{e}}{G_\alpha} \geq x\Big) = P(\mathfrak{e} \geq xG_\alpha) = E(e^{-xG_\alpha})$$

i) There exists a $(1-\alpha,\, K_\alpha)$ *positive compound Poisson process* $(Y_t,\, t \geq 0)$ *such that :*

$$\Delta_\alpha \overset{\text{(law)}}{=} \int_0^\infty e^{-t}\, dY_t \tag{1.21}$$

ii) Δ_α *satisfies the affine equation :*

$$\Delta_\alpha \overset{\text{(law)}}{=} U^{1/1-\alpha}(\Delta_\alpha + K_\alpha) \tag{1.22}$$

where, on the RHS of (1.22), U, Δ_α *and* K_α *are assumed independent, and* U *is uniformly distributed on* $[0,1]$.

We note that decompositions such as (1.22), and below (1.69), were also studied in Jurek [J].

1.4 Some properties of the r.v.'s G_α $(0 \leq \alpha \leq 1)$. Recall that, for any $\alpha \in]0,1[$, the r.v. G_α is defined either via its density (1.17) or via its Stieltjes transform (1.18) $\big($or (1.19)$\big)$.

Theorem 1.2

1) The law of $G_{1/2}$ *is beta* $\left(\frac{1}{2}, \frac{1}{2}\right)$, *i.e.* $G_{1/2}$ *is arc-sine distributed :*

$$f_{G_{1/2}}(u) = \frac{1}{\pi} \frac{1}{\sqrt{u(1-u)}}\, 1_{[0,1]}(u) \tag{1.23}$$

2) Let $p \geq 2$ *denote an integer, and let* $B_1, \dots, B_{p-1}$ *be a sequence of* $(p-1)$ *independent variables, such that for any* $i = 1, 2, \dots, p-1$, B_i *is distributed as beta* $\left(\frac{i}{p}, 1 - \frac{i}{p}\right)$. *Let* ε_p *denote a variable which is uniformly distributed on* $\big\{1, 2, \dots, (p-1)\big\}$, *and is independent of the sequence* $\big((B_i), i = 1, \cdots, p-1\big)$.
Then, for $\alpha = 1/p$, *one has :*

$$G_\alpha = G_{1/p} \overset{\text{(law)}}{=} B_{\varepsilon_p} \tag{1.24}$$

i.e. :

$$f_{G_{1/p}}(u) = \frac{1}{\pi(p-1)} \sum_{i=1}^{p-1} \sin\left(\frac{\pi i}{p}\right) u^{\frac{i}{p}-1}(1-u)^{-\frac{i}{p}}\, 1_{[0,1]}(u) \tag{1.25}$$

$$= \frac{1}{\pi(p-1)} \sum_{i=1}^{p-1} \sin\left(\frac{\pi i}{p}\right) u^{-\frac{i}{p}}(1-u)^{\frac{i}{p}-1}\, 1_{[0,1]}(u) \tag{1.26}$$

3) $$G_\alpha \overset{\text{(law)}}{=} 1 - G_\alpha \tag{1.27}$$

4) As $\alpha \longrightarrow 1$, G_α converges in law to a r.v. we denote as G_1, which is uniformly distributed on $[0,1]$.

5) As $\alpha \longrightarrow 0$, G_α converges in law to a r.v. we denote as G_0, which satisfies :

$$\begin{aligned} i) \quad f_{G_0}(u) &= \frac{1}{\pi}\left(\int_0^1 (\sin \pi\beta) u^{\beta-1}(1-u)^{-\beta} d\beta\right) 1_{[0,1]}(u) \\ &= \frac{1}{u(1-u)} \, \frac{1}{\pi^2 + \left(\log\left(\frac{1-u}{u}\right)\right)^2} \, 1_{[0,1]}(u) \end{aligned} \tag{1.28}$$

$$ii) \quad G_0 \overset{(\text{law})}{=} \frac{1}{1+\exp \pi C} \tag{1.29}$$

where C is a standard Cauchy r.v.

iii) The Stieltjes transform of (the law of) G_0 is given by :

$$S(f_{G_0})(\lambda) := \int_0^1 \frac{f_{G_0}(u)}{\lambda+u} du = E\Big(\frac{1}{\lambda+G_0}\Big) = \frac{1}{\lambda(1+\lambda)} \, \frac{1}{\log\left(\frac{1+\lambda}{\lambda}\right)} \quad (\lambda \geq 0) \tag{1.30}$$

1.5 The variables G_α, the unilateral stable laws and the Mittag-Leffler distributions.

Let $\mu \in]0,1[$. We denote by T_μ a unilateral ($\mathbb{R}_+-$valued) stable r.v. with parameter μ :

$$E(e^{-\lambda T_\mu}) = \exp(-\lambda^\mu) \quad (\lambda \geq 0) \tag{1.31}$$

Let T'_μ be an independent copy of T_μ, and define :

$$Z_\mu :\overset{(\text{law})}{=} \left(\frac{T_\mu}{T'_\mu}\right)^\mu \tag{1.32}$$

On the other hand, we denote by M_μ a r.v. distributed with the Mittag-Leffler law of index μ, that is (see [CY], p. 114) :

$$E\big[\exp(\lambda M_\mu)\big] = \sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma(n\mu+1)} \quad (\lambda \in \mathbb{R}) \tag{1.33}$$

and, consequently :

$$E(M_\mu^n) = \frac{\Gamma(n+1)}{\Gamma(\mu n+1)} \quad (n > -1) \tag{1.34}$$

from which we deduce :

$$M_\mu \overset{(\text{law})}{=} \frac{1}{(T_\mu)^\mu} \tag{1.35}$$

There exists a remarkable link between the variables G_α and $Z_{1-\alpha}$.

Theorem 1.3

1) (Lamperti [Lamp]) The variable Z_μ has the density :

$$f_{Z_\mu}(x) := \frac{\sin(\pi\mu)}{\pi\mu} \frac{1}{x^2 + 2x\cos(\pi\mu) + 1} \mathbf{1}_{x\geq 0} \tag{1.36}$$

2) For any $\alpha \in (0,1)$,

$$i) \quad G_\alpha \overset{\text{(law)}}{=} \frac{(Z_{1-\alpha})^{1/\alpha}}{1 + (Z_{1-\alpha})^{1/\alpha}} \overset{\text{(law)}}{=} \frac{(T_{1-\alpha})^{\frac{1-\alpha}{\alpha}}}{(T'_{1-\alpha})^{\frac{1-\alpha}{\alpha}} + (T_{1-\alpha})^{\frac{1-\alpha}{\alpha}}} \tag{1.37}$$

(This relation implies obviously that $G_\alpha \overset{\text{(law)}}{=} 1 - G_\alpha$)

$$ii) \quad G_\alpha \overset{\text{(law)}}{=} \frac{(M_{1-\alpha})^{1/\alpha}}{(M_{1-\alpha})^{1/\alpha} + (M'_{1-\alpha})^{1/\alpha}} \tag{1.38}$$

where, on the RHS of (1.38) $M_{1-\alpha}$ and $M'_{1-\alpha}$ are two independent copies of Mittag-Leffler r.v's of index $1 - \alpha$.

1.6 The "algebra" of the variables γ_α, G_α, and $X_{a,b}$.

It is a classical result that, if γ_a and γ_b denote two independent gamma variables with respective parameters a and b, then :

$$\left(\frac{\gamma_a}{\gamma_a + \gamma_b}, \; \gamma_a + \gamma_b\right) \overset{\text{(law)}}{=} (\beta_{a,b}, \gamma_{a+b}) \tag{1.39}$$

where, on the RHS of (1.39), $\beta_{a,b}$ and γ_{a+b} are independent and distributed respectively as beta (a,b) and gamma $(a+b)$. From this relation, we deduce, in particular :

$$\gamma_{a+b} \cdot \beta_{a,b} \overset{\text{(law)}}{=} \gamma_a \quad \text{and, if } b = 1 - a, \quad \mathfrak{e} \cdot \beta_{a,1-a} \overset{\text{(law)}}{=} \gamma_a \tag{1.40}$$

It is the kind of properties such as (1.39) and (1.40) which justifies the usual terminology of "beta-gamma algebra" (see also Dufresne [Duf] for further developments). Our r.v.'s G_α $(0 \leq \alpha \leq 1)$ also enjoy - together with the r.v.'s $X_{a,b}$ defined below - some "algebraic properties" akin to those of the beta-gamma algebra. We note in fact that, for $p \geq 2$, p an integer, and $\alpha = \frac{1}{p}$ the density of G_α is a barycentric combination of some beta densities, as asserted by Theorem 1.2.

Theorem 1.4

1) Existence of the variables $X_{a,b}$. For every a, b such that : $0 < a \leq b \leq 1$, there exists an $\mathbb{R}_+$-valued variable $X_{a,b}$ such that :

$$E\big[\exp(-\lambda X_{a,b})\big] = \left(\frac{b}{a}\right) \frac{(1+\lambda)^a - 1}{(1+\lambda)^b - 1} \quad (\lambda \geq 0) \tag{1.41}$$

2) These variables $X_{a,b}$ are infinitely divisible and satisfy :
for any sequence : $0 < a_1 \leq a_2 \leq \ldots \leq a_n < 1$:

$$X_{a_1,a_n} \overset{\text{(law)}}{=} \sum_{i=1}^{n-1} X_{a_i,a_{i+1}} \quad ; \quad X_{a,a} = 0 \tag{1.42}$$

where on the RHS of (1.42), the r.v.'s are assumed independent.

3) Algebra properties. For any α, $0 \leq \alpha \leq 1$:

$$\mathfrak{e} \overset{\text{(law)}}{=} \mathfrak{e}_1 G_\alpha + \mathfrak{e}_2 G_{1-\alpha} \tag{1.43}$$

where, on the RHS of (1.43), $\mathfrak{e}_1, \mathfrak{e}_2, G_\alpha$ and $G_{1-\alpha}$ are independent, and $\mathfrak{e}, \mathfrak{e}_1, \mathfrak{e}_2$ are standard exponential variables. In other terms, the variables G_α and $G_{1-\alpha}$ yield an affine decomposition of the exponential law.

4) More generally, for any $\alpha \in \left[\frac{1}{2}, 1\right]$:

$$\mathfrak{e} G_\alpha \overset{\text{(law)}}{=} \gamma_{(1-\alpha)} + X_{1-\alpha,\alpha} \tag{1.44}$$

where as usual, the r.v.'s which appear on each side of (1.44) are assumed independent, whereas for $\alpha \in \left[0, \frac{1}{2}\right]$:

$$X_{\alpha,1-\alpha} + \mathfrak{e} G_\alpha \overset{\text{(law)}}{=} \gamma_{(1-\alpha)} \tag{1.45}$$

We note that (1.44) implies that, for $\alpha \geq \dfrac{1}{2}$, $\mathfrak{e} G_\alpha$ is infinitely divisible, and that the addition term by term of (1.44) and (1.45), where α is replaced by $(1-\alpha)$, implies (1.43).

1.7 The r.v.'s. $G_{\alpha,\beta}$ and their "algebraic" properties. $(0 < \alpha, \beta < 1)$.

Recall that the (laws of) G_α $(0 < \alpha < 1)$ are characterized by :

$$E\left[\frac{1}{1+\lambda G_\alpha}\right] = \left(\frac{\alpha}{1-\alpha}\right) \frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha - 1} \quad (\lambda \geq 0) \tag{1.46}$$

This relation led us to raise the following questions :

- Does there exist variables $G_{\alpha,\beta}$ such that :

$$E\left[\frac{1}{1+\lambda G_{\alpha,\beta}}\right] = \frac{\alpha}{1-\beta} \, \frac{1-(1+\lambda)^{\beta-1}}{(1+\lambda)^\alpha - 1} \quad ? \tag{1.47}$$

- If yes, do these variables possess "algebraic" properties similar to those described in the above Theorem 1.4 ?

  The next Theorem answers these questions in the affirmative.

Theorem 1.5

Let α, β such that : $0 < \alpha, \beta < 1$.

1) (Existence of the variable $G_{\alpha,\beta}$)

i) There exists a r.v. $G_{\alpha,\beta}$, taking values in $[0,1]$, such that :

$$E[e^{-\lambda \mathfrak{e} G_{\alpha,\beta}}] = E\left[\frac{1}{1+\lambda G_{\alpha,\beta}}\right] = \frac{\alpha}{1-\beta}\,\frac{1-(1+\lambda)^{\beta-1}}{(1+\lambda)^{\alpha}-1} \quad (\lambda \geq 0) \tag{1.48}$$

ii) In close relation with (1.48), the Stieltjes transform of $G_{\alpha,\beta}$, is :

$$E\left[\frac{1}{\lambda+G_{\alpha,\beta}}\right] = \frac{\alpha}{1-\beta}\,\frac{(\lambda^{\beta-1}-(1+\lambda)^{\beta-1})\lambda^{\alpha-\beta}}{(1+\lambda)^{\alpha}-\lambda^{\alpha}} \quad (\lambda \geq 0) \tag{1.49}$$

iii) The density of $G_{\alpha,\beta}$, denoted by $f_{G_{\alpha,\beta}}$ is :

$$f_{G_{\alpha,\beta}}(u) = 1_{[0,1]}(u)\cdot \tag{1.50}$$
$$\frac{\alpha}{\pi(1-\beta)}\cdot\frac{(1-u)u^{\alpha-1}\sin(\pi\alpha)+u^{2\alpha-\beta}(1-u)^{\beta-1}\sin(\pi\beta)+(1-u)^{\alpha+\beta-1}u^{\alpha-\beta}\sin\big(\pi(\alpha-\beta)\big)}{(1-u)^{2\alpha}-2(1-u)^{\alpha}u^{\alpha}\cos(\pi\alpha)+u^{2\alpha}}$$

(note that it is not quite obvious to verify that $f_{G_{\alpha,\beta}} \geq 0$, for $\alpha < \beta$).

$$iv) \quad G_{\alpha,\alpha} \overset{\text{(law)}}{=} G_{\alpha} \tag{1.51}$$

$$v) \quad G_{\alpha,1-\alpha} \quad \text{is a beta } (\alpha, 1-\alpha) \text{ r.v.} \tag{1.52}$$

2) Algebraic properties :

$$i) \quad \text{if} \quad \alpha+\beta \geq 1, \quad \text{then} \quad \mathfrak{e} G_{\alpha,\beta} \overset{\text{(law)}}{=} \gamma_{1-\beta} + X_{1-\beta,\alpha} \tag{1.53}$$

$$ii) \quad \text{if} \quad \alpha+\beta \leq 1, \quad \text{then} \quad \gamma_{(1-\beta)} \overset{\text{(law)}}{=} \mathfrak{e} G_{\alpha,\beta} + X_{\alpha,1-\beta} \tag{1.54}$$

$$iii) \quad \text{for all} \quad 0 < \alpha, \beta, \gamma < 1 \ : \ \mathfrak{e}_1 G_{\alpha,\beta} + \mathfrak{e}_2 G_{\beta,\gamma} \overset{\text{(law)}}{=} \mathfrak{e}_1 G_{\alpha,\gamma} + \mathfrak{e}_2 G_{\beta} \tag{1.55}$$

and, if $\alpha+\beta \geq 1$, from (1.55) and (1.53)

$$\gamma_{(1-\beta)} + X_{1-\beta,\alpha} + \mathfrak{e}_2 G_{\beta,\gamma} \overset{\text{(law)}}{=} \mathfrak{e}_1 G_{\alpha,\gamma} + \mathfrak{e}_2 G_{\beta} \tag{1.56}$$

whereas, if $\alpha+\beta \leq 1$, then, from (1.55) and (1.54) :

$$\gamma_{(1-\beta)} + \mathfrak{e} G_{\beta,\gamma} \overset{\text{(law)}}{=} \mathfrak{e}_1 G_{\alpha,\gamma} + \mathfrak{e}_2 G_{\beta} + X_{\alpha,1-\beta} \tag{1.57}$$

iv) if $0 < \alpha < \beta < 1$, then $\mathfrak{e}(1-G_{\alpha,\beta}) \overset{\text{(law)}}{=} \gamma_{\beta-\alpha} + \mathfrak{e}\, G_{\alpha,\beta}$

if $0 < \beta < \alpha < 1$, then $\gamma_{\alpha-\beta} + \mathfrak{e}(1-G_{\alpha,\beta}) \overset{\text{(law)}}{=} \mathfrak{e}\, G_{\alpha,\beta}$

Of course, in all the above relations, on each side, the featured r.v.'s are independent. The relations (1.43), (1.44) and (1.45) are particular cases of the above relations (1.53), ..., (1.57).

1.8 On (δ, G) self-decomposable variables.

The formula (1.16), where we do not mention the index α :

$$E(e^{-\lambda\Delta}) = \exp\Big(-\delta\int_0^\infty (1-e^{-\lambda x})\frac{dx}{x}\,E(e^{-xG})\Big) \tag{1.58}$$

led us to study the r.v.'s Δ whose laws may be obtained from those of G via the relation (1.58), thus generalizing the relation between Δ_α and G_α.

Remark and a definition : Let G be an $\mathbb{R}_+-$valued r.v. The following properties are equivalent :

$$i)\quad \int_1^\infty \frac{dx}{x}E(e^{-xG}) < \infty \tag{1.59}$$

$$ii)\quad E\Big(\log^+\big(\frac{1}{G}\big)\Big) < \infty \tag{1.60}$$

$$iii)\qquad \int_0^\infty (x\wedge 1)\,\frac{dx}{x}E(e^{-xG}) < \infty,$$

$$\text{i.e. : the measure } \frac{dx}{x}E(e^{-xG})\,1_{x\geq 0}\quad \text{is the Lévy measure of a subordinator} \tag{1.61}$$

$$iv)\quad E\Big(\log\big(1+\frac{\lambda}{G}\big)\Big) < \infty \quad \text{for some (hence all) } \lambda > 0 \tag{1.62}$$

Let G satisfy one (hence all) of these conditions, and let $\delta > 0$. We say that a r.v. Δ is (δ, G) self-decomposable if :

$$E(e^{-\lambda\Delta}) = \exp\Big(-\delta\int_0^\infty (1-e^{-\lambda x})\frac{dx}{x}\,E(e^{-xG})\Big)\quad (\lambda\geq 0) \tag{1.63}$$

$$= \exp\Big(-\delta\,E\big(\log\big(1+\frac{\lambda}{G}\big)\big)\Big)\quad (\lambda\geq 0) \tag{1.64}$$

$\big($Note that (1.63) may be considered as a definition of the law of Δ in terms of (δ, G), whereas (1.64) follows from (1.63) via the simple Frullani integral argument $\big($see, e.g., Lebedev [Leb] p.6$\big)$.

The (δ, G) self-decomposable r.v.'s are closely linked to the standard gamma subordinator; in fact, their laws are the generalized Gamma convolutions which have been studied extensively by L. Bondesson ([B1], [B2]).

Theorem 1.6

Let $(\gamma_t, t\geq 0)$ denote the gamma standard subordinator, i.e. the subordinator such that :

$$E(e^{-\lambda\gamma_t}) = \frac{1}{(1+\lambda)^t} = \exp\big(-t\log(1+\lambda)\big)\quad (t,\lambda\geq 0)$$

and let $h :]0,\infty[\to\mathbb{R}_+$, a Borel function.

1) Let

$$\Delta_h := \int_0^\infty h(u) d\gamma_u \tag{1.65}$$

Then Δ_h is finite a.s. if and only if :

$$\int_0^\infty \log\big(1 + h(u)\big) du < \infty \tag{1.66}$$

2) Under the hypothesis (1.66), Δ_h is self-decomposable and :

$$\begin{aligned} E(e^{-\lambda\Delta_h}) \quad &= \quad \exp\Big(-\int_0^\infty (1 - e^{-\lambda x})\; F_h(x)\frac{dx}{x}\Big) \\ \textit{with} \qquad & F_h(x) := \int_0^\infty e^{-\frac{x}{h(u)}} du \end{aligned}$$

3) For all positive r.v. G satisfying (1.59) and all $\delta > 0$, there exists h satisfying (1.66) so that :

$$\delta E(e^{-xG}) = F_h(x) = \int_0^\infty e^{-\frac{x}{h(u)}} du \tag{1.67}$$

In other terms, all r.v.'s Δ which are (δ, G) self-decomposable can be written as $\Delta \overset{\text{(law)}}{=} \Delta_h$ because, by (1.67):

$$E(e^{-\lambda\Delta_h}) = \exp\Big(-\delta\int_0^\infty (1 - e^{-\lambda x})\; E(e^{-xG})\frac{dx}{x}\Big)$$

Here are some further precisions about this theorem :

- An explicit relation between h and G as in (1.67) is :

  $$h(u) = \frac{1}{\mathcal{G}^{-1}\left(\frac{u}{\delta}\right)}\,, \text{ for } u \in (0, \delta)\,, \text{ and } 0, \text{ for } u > \delta$$

  where $\mathcal{G}^{-1}$ denotes the inverse (in the sense of the composition of functions) of the distribution function of the r.v. G.

- Moreover, it is known (see [B2]; see also [SVH], Theorem 5.24, p. 362) that a positive r.v. $\Delta$is of the form $\Delta_h = \int_0^\infty h(u) d\gamma(u)$, i.e : its law is a generalized gamma convolution if, and only if, its Laplace transform $\psi_\Delta(\lambda) := E(e^{-\lambda\Delta})$is hyperbolically completely monotone, i.e : it satisfies

  $$\forall u > 0\,, \text{ the function : } \left(v + \frac{1}{v}\right) \to \psi_\Delta(uv)\psi_\Delta\left(\frac{u}{v}\right)$$

  is completely monotone, as a function of $\left(v + \frac{1}{v}\right)$.

Theorem 1.7
Let G satisfy (1.60), and let Δ denote a r.v. which is (δ, G) self-decomposable.

1) *There exists a (δ, K) positive compound Poisson process $(Y_t,\ t \geq 0)$ with $K \overset{(\text{law})}{=} \mathfrak{e}/G$, such that :*

$$\Delta := \int_0^\infty e^{-t} dY_t \tag{1.68}$$

2) *Δ satisfies the affine equation :*

$$\Delta \overset{(\text{law})}{=} U^{1/\delta}(\Delta + K) \tag{1.69}$$

where on the RHS of (1.69), the r.v.'s. U, Δ and K are independent, and U is uniform on $[0,1]$.

3) *Let $\psi(\lambda) := E(e^{-\lambda\Delta})$. Then, the Stieltjes transform of G equals :*

$$E\Big(\frac{1}{\lambda + G}\Big) = -\frac{\psi'}{\psi}(\lambda) = -\frac{\partial}{\partial\lambda}\big(\log E(e^{-\lambda\Delta})\big) \quad (\lambda \geq 0) \tag{1.70}$$

We note that Theorem 1.7 presents the points 3) and 4) of Theorem 1.1 in a more general set-up. We shall now establish a converse to Theorem 1.7 which, essentially, hinges upon the properties of the inverse Stieltjes transform. This leads to the following :

<u>Definition</u> : A function $F:]0,\infty[\longrightarrow]0,\infty[$, which is C^1, is said to satisfy condition (ST, δ) (obviously, ST stands for Stieltjes transform) if :

i) F extends holomorphically to $\mathbb{C} \setminus]-\infty, 0[$;

ii) For any $u \geq 0$, $F_+(u) := \lim\limits_{\eta \to 0+} F(-u + i\eta)$, resp : $F_-(u) := \lim\limits_{\eta \to 0+} F(-u - i\eta)$ exists, is continuous, and satisfies :

$$\mathrm{Im}\,\big(F_-(u) - F_+(u)\big) \geq 0, \text{ for any } \quad u \geq 0 \tag{1.71}$$

iii) $\lim\limits_{\substack{\lambda \longrightarrow \infty \\ \lambda \in \mathbb{R}}} \lambda F(\lambda) = \delta$

This definition proves useful in the following :

Theorem 1.8
Let Δ denote a positive r.v. with Laplace transform ψ, i.e. $E[e^{-\lambda\Delta}] = \psi(\lambda)$ $(\lambda \geq 0)$. Assume that $F := -\dfrac{\psi'}{\psi}$ satisfies the condition (ST, δ). Then :

$$f(u) := \frac{1}{2\pi\delta}\,\mathrm{Im}\big(F_-(u) - F_+(u)\big) \quad (u \geq 0)$$

defines a density of probability on $\mathbb{R}_+$, and Δ is a r.v. which is (δ, G) self-decomposable, when G denotes a r.v. with density $f_G = f$.

Acknowledgments :

1) After making some quite informal computation, we were able to "verify" that the density of G_α, resp. $G_{\alpha,\beta}$, is indeed given by (1.17), resp. (1.50). We thank most sincerely Mrs Y. Yano who indicated to us a more rigorous proof based on Stieltjes transform inversion.

2) We also thank Prof. Z.J. Jurek who provided us with some references, and also helped us to eradicate some misprints.

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2 Proof of Theorem 1.1

2.1 We begin with point 1) of Theorem 1.1, which we now recall :

Theorem 1.1, point 1)

i) $$\Delta_\alpha \overset{\text{(law)}}{=} \frac{\gamma_{(1-\alpha)}}{\beta_{(\alpha,1)}} \tag{2.1}$$

where on the RHS of (2.1) , $\gamma_{(1-\alpha)}$ and $\beta_{(\alpha,1)}$ are respectively two independent gamma $(1-\alpha)$ and beta $(\alpha,1)$ variables.

ii) The density f_{Δ_α}, of Δ_α, is given by :

$$f_{\Delta_\alpha}(x) := \frac{\alpha}{\Gamma(1-\alpha)} x^{-\alpha-1}(1-e^{-x})1_{[0,\infty[}(x) \tag{2.2}$$

iii) The Laplace transform of (the law of) Δ_α is :

$$E(e^{-\lambda\Delta_\alpha}) = (1+\lambda)^\alpha - \lambda^\alpha \quad (\lambda \geq 0) \tag{2.3}$$

As indicated in the Introduction, this point is a particular case of the results of M. Winkel $\big($[Wink]$\big)$. However, below, we give three proofs of this point . The two first proofs are very specific to the Bessel process context in which we are working whereas the third one, of a more general kind, uses arguments close to those of M. Winkel.

2.1.1 First proof of point 1) of Theorem 1.1 :

2.1.1.a) By scaling, we have :

$$\Delta_\alpha \overset{\text{(law)}}{=} \mathfrak{e}(d_1 - g_1) \overset{\text{(law)}}{=} \mathfrak{e}\big((1-g_1) + (d_1 - 1)\big) \tag{2.4}$$

Furthermore, $(1-g_1, d_1-1) \overset{\text{(law)}}{=} (1-g_1, R_1^2 T_0^{(1)})$ where the pair $(1-g_1, R_1)$ is independent from $T_0^{(1)} \equiv \inf\{t \geq 0 : R_t^{(1)} = 0\}$, with $(R_u^{(1)},\, u \geq 0)$ a Bessel process starting from 1. This is obtained by applying the Markov property to R at time 1, together with the scaling property. It is well-known $\big($see, e.g. [D-M,R,V,Y], [Y] p. 14, [Get]$\big)$ that :

$$T_0^{(1)} \overset{\text{(law)}}{=} 1/2\gamma_{(\alpha)} \tag{2.5}$$

where $\gamma_{(\alpha)}$ is gamma (α) distributed. Thus, from (2.4), we get :

$$\Delta_\alpha \overset{(\text{law})}{=} \mathfrak{e}\Big((1-g_1)+R_1^2\,\frac{1}{2\gamma_{(\alpha)}}\Big) \tag{2.6}$$

where, on the RHS, the pair : $\big((1-g_1),R_1\big)$ is independent from $\gamma_{(\alpha)}$. Morever, classical properties of the Bessel meander $\big($see, e.g., [D-M,R,V,Y], where these properties are recalled$\big)$ imply :

$$(R_1^2,1-g_1) \overset{(\text{law})}{=} \big((1-g_1)2\mathfrak{e}_1,(1-g_1)\big) \tag{2.7}$$

where $\mathfrak{e}_1$ is a standard exponential variable, independent from g_1, and g_1 is beta $(\alpha,1-\alpha)$ distributed. Bringing (2.7) in (2.6), we obtain :

$$\Delta_\alpha \overset{(\text{law})}{=} (1-g_1)\mathfrak{e}\Big(1+\frac{\mathfrak{e}_1}{\gamma_{(\alpha)}}\Big)$$

where, on the RHS, the 4 r.v.'s $g_1,\mathfrak{e},\mathfrak{e}_1,\gamma_{(\alpha)}$ are assumed independent. Furthermore, the classical properties of the "beta-gamma algebra" imply :

$$(1-g_1)\mathfrak{e} \overset{(\text{law})}{=} \gamma_{(1-\alpha)} \quad \text{and} \quad 1+\frac{\mathfrak{e}_1}{\gamma_{(\alpha)}} \overset{(\text{law})}{=} \frac{1}{\beta_{(\alpha,1)}}$$

hence, finally :

$$\Delta_\alpha \overset{(\text{law})}{=} \frac{\gamma_{(1-\alpha)}}{\beta_{(\alpha,1)}}$$

2.1.1.b) The expression of the density $\big($given by (2.2)$\big)$ of Δ_α follows from (2.1). Furthermore :

$$\begin{aligned} E(e^{-\lambda\Delta_\alpha}) &= E\Big(e^{-\lambda\frac{\gamma_{(1-\alpha)}}{\beta_{(\alpha,1)}}}\Big) = E\Big(\frac{1}{1+\frac{\lambda}{\beta_{(\alpha,1)}}}\Big)^{1-\alpha} = E\Big(\frac{\beta_{(\alpha,1)}}{\lambda+(\beta_{\alpha,1})}\Big)^{1-\alpha} \\ &= \alpha\int_0^1\Big(\frac{u}{\lambda+u}\Big)^{1-\alpha}u^{\alpha-1}du = \alpha\int_0^1(\lambda+u)^{\alpha-1}du = (1+\lambda)^\alpha-\lambda^\alpha \end{aligned}$$

2.1.2 <u>Second proof of point 1) of Theorem 1.1</u> :
It hinges upon the same arguments as in the preceding proof, but it has a more analytic flavor. We shall show that :

$$E(e^{-\lambda\Delta_\alpha}) = (1+\lambda)^\alpha-\lambda^\alpha \quad (\lambda\geq 0) \tag{2.8}$$

We denote by $P^{(\alpha)}$ the distribution of the Bessel process, starting from 0, with dimension $d=2(1-\alpha)$ $(0<\alpha<1)$ and let $(A_t:=t-g_t,\ t\geq 0)$ denote the age process of excursions of R away from 0. Then, for fixed $t\geq 0$, one has :

$$E^{(\alpha)}[e^{-\lambda(d_t-g_t)}] = E^{(\alpha)}(e^{-\lambda(A_t+T_0\circ\theta_t)}) \tag{2.9}$$

(where T_0 denotes the first hitting time of 0 by $(R_t, t \geq 0)$ and $(\theta_t,\, t \geq 0)$ is the usual family of translation operators.)

$$= E^{(\alpha)}\big(e^{-\lambda A_t} E^{(\alpha)}_{R_t}(e^{-\lambda T_0})\big) \tag{2.10}$$

The Laplace transform of T_0 featured in (2.10) may be computed explicitly $\big($see e.g. [Get]$\big)$, in agreement with (2.5) :

$$\begin{aligned} E^{(\alpha)}[e^{-\lambda(d_t-g_t)}] &= E^{(\alpha)}\big(e^{-\lambda A_t} K_\alpha(R_t\sqrt{2\lambda})\,(R_t\sqrt{2\lambda})^\alpha\big) && (2.11)\\ &\quad \text{(where } K_\alpha \text{ denotes the Bessel-Mac Donald function with index } \alpha) \\ &= E^{(\alpha)}\Big[e^{-\lambda A_t}\big(\Phi(1,1-\alpha,\lambda A_t) - \Gamma(1-\alpha)(\lambda A_t)^\alpha e^{\lambda A_t}\big)\Big] && (2.12) \end{aligned}$$

where $\Phi(1, 1-\alpha, \cdot)$ denotes the confluent hypergeometric function with parameter $(1, 1-\alpha)$ $\big($see [Leb], p. 260$\big)$. We now replace in (2.12) the fixed time t by a variable $\mathfrak{e}$, exponentially distributed, and independent from $(R_u,\, u \geq 0)$. Note that, by scaling :

$$A_{\mathfrak{e}} \overset{\text{(law)}}{=} \mathfrak{e}\, A_1 \overset{\text{(law)}}{=} \mathfrak{e}\, \beta_{(1-\alpha,\alpha)} \overset{\text{(law)}}{=} \gamma_{(1-\alpha)} \tag{2.13}$$

hence, we get :

$$\begin{aligned} & E(e^{-\lambda\Delta_\alpha}) \\ =\; & E^{(\alpha)}(e^{-\lambda(d_{\mathfrak{e}}-g_{\mathfrak{e}})}) \\ =\; & \frac{1}{\Gamma(1-\alpha)}\int_0^\infty e^{-\lambda z - z} z^{-\alpha}\Phi(1,1-\alpha,\lambda z)dz - \int_0^\infty e^{-z}(\lambda z)^\alpha z^{-\alpha}dz \\ =\; & \left[\frac{1}{\Gamma(1-\alpha)}\int_0^\infty e^{-(\lambda+1)z} z^{-\alpha}\left(\sum_{k=0}^\infty \frac{k!\Gamma(1-\alpha)}{\Gamma(1-\alpha+k)}\;\frac{(\lambda z)^k}{k!}\right)dz\right] - \lambda^\alpha \\ & \text{(from the definition of the hypergeometric function } \Phi(1,1-\alpha,\cdot)) \\ =\; & \Big\{\sum_{k=0}^\infty \int_0^\infty dz\; e^{-(\lambda+1)z}\frac{\lambda^k z^{k-\alpha}}{\Gamma(1-\alpha+k)}\Big\} - \lambda^\alpha \\ =\; & \Big\{\sum_{k=0}^\infty \frac{\lambda^k}{\Gamma(1-\alpha+k)}\int_0^\infty \frac{e^{-u}u^{k-\alpha}}{(1+\lambda)^{k-\alpha+1}}du\Big\} - \lambda^\alpha \\ =\; & \sum_{k=0}^\infty \left(\frac{\lambda}{1+\lambda}\right)^k \frac{1}{(1+\lambda)^{1-\alpha}} - \lambda^\alpha = (1+\lambda)^{\alpha-1}\frac{1}{1-\frac{\lambda}{1+\lambda}} - \lambda^\alpha = (1+\lambda)^\alpha - \lambda^\alpha. \end{aligned}$$

2.1.3 A third proof of point 1) of Theorem 1 :
It hinges only - as in the proof of M. Winkel [Wink] - upon the fact that the process :

$$\tau_l := \inf\{t \geq 0\,;\; L_t > l\}\,,\; l \geq 0$$

is a stable subordinator, without drift term, where $(L_t,\, t \geq 0)$ denotes the local time process at 0 of $(R_t,\, t \geq 0)$. Thus :

$$E(e^{-\lambda\tau_l}) = \exp\Big\{-l\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\,2^{-\alpha}\lambda^\alpha\Big\} := e^{-l\Phi(\lambda)} \qquad (\lambda \geq 0) \tag{2.14}$$

where $\Phi(\lambda)$ is the characteristic exponent of $(\tau_l,\, l \geq 0)$ $\big($cf, [D-M,R,V,Y]$\big)$ for a discussion of the values of normalization constants related to $(L_t,\, t \geq 0)$ and $(\tau_l,\, l \geq 0)\big)$.
Now, let in general, $(\tau_l,\, l \geq 0)$ denote a subordinator without drift. In other terms :

$$\begin{aligned} & E(e^{-\lambda\tau_l}) = \exp\big(-l\Phi(\lambda)\big) \\ \text{with} \quad & \Phi(\lambda) := \exp - \int_0^\infty (1-e^{-\lambda x})\nu(dx) \end{aligned} \tag{2.15}$$

where ν denotes the Lévy measure of $(\tau_l,\, l \geq 0)$. Let us define :

$$L_t := \inf\{l\ ;\ \tau_l > t\}\,,\ t \geq 0$$

and let $\mathfrak{e}$ denote an exponential variable, with mean 1, independent from $(\tau_l,\, l \geq 0)$.

Lemma 2.1. *Let*

$$\Delta^{(\tau)} := \tau_{(L_\mathfrak{e})} - \tau_{(L_\mathfrak{e})^-} \tag{2.16}$$

Then

$$E(e^{-\lambda\Delta^{(\tau)}}) = \frac{\Phi(1+\lambda)-\Phi(\lambda)}{\Phi(1)} \tag{2.17}$$

Clearly, point *iii*) of our Theorem 1 is an immediate consequence of (2.17), when Lemma 2.1. is applied to the subordinator defined by (2.14), i.e. when $\Phi(\lambda) = \dfrac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} 2^{-\alpha}\lambda^\alpha \quad (\lambda \geq 0)$.

Proof of Lemma 2.1. : By definition of $\Delta^{(\tau)}$, we have :

$$\begin{aligned} E(e^{-\lambda\Delta^{(\tau)}}) &= E\Big(\int_0^\infty e^{-t-\lambda(\tau_{L_t}-\tau_{(L_t)^-}}dt\Big) \\ &= E\Big(\sum_{l>0}\int_{\tau_{l^-}}^{\tau_l} e^{-t}e^{-\lambda\delta_l}dt\Big) \quad (\text{where } \delta_l := \tau_l - \tau_{l^-}) \\ &= E\Big(\sum_{l>0}(e^{-\tau_{l^-}} - e^{-\tau_l})e^{-\lambda\delta_l}\Big) \\ &= E\Big(\int_0^\infty dl\, e^{-\tau_{l^-}} \int_0^\infty (1-e^{-v})e^{-\lambda v}\nu(dv)\Big) \\ &= E\Big(\int_0^\infty e^{-\tau_l}dl\Big)\big(\Phi(1+\lambda)-\Phi(\lambda)\big) \\ &= \big(\Phi(1+\lambda)-\Phi(\lambda)\big)\int_0^\infty e^{-l\Phi(1)}dl = \frac{\Phi(1+\lambda)-\Phi(\lambda)}{\Phi(1)} \qquad \blacksquare \end{aligned}$$

2.2 Proof of point 2) of Theorem 1.1 :

We first recall this point 2) :

i) Δ_α *is self-decomposable and the Lévy-Khintchine formula writes :*

$$E(e^{-\lambda\Delta_\alpha}) = \exp\Big(-(1-\alpha)\int_0^\infty (1-e^{-\lambda x})\,\frac{dx}{x}\,E(e^{-xG_\alpha})\Big) \tag{2.18}$$

where G_α *denotes a r.v. taking values in* $[0,1]$*, with density :*

$$f_{G_\alpha}(u) = \frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi}\ \frac{u^{\alpha-1}(1-u)^{\alpha-1}}{(1-u)^{2\alpha} - 2(1-u)^\alpha u^\alpha\cos(\pi\alpha) + u^{2\alpha}}\, 1_{[0,1]}(u) \tag{2.19}$$

ii) The law of G_α is characterized by its Stieltjes transform :

$$\begin{aligned} S(f_{G_\alpha})(\lambda) : &= \int_0^1 \frac{f_{G_\alpha}(u)}{\lambda+u} du = E\Big(\frac{1}{\lambda+G_\alpha}\Big) \\ &= \frac{\alpha}{1-\alpha} \; \frac{\lambda^{\alpha-1}-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-\lambda^\alpha} \quad (\lambda \geq 0) \end{aligned} \tag{2.20}$$

or, equivalently :

$$E(e^{-\lambda \mathfrak{e} G_\alpha}) = E\Big(\frac{1}{1+\lambda G_\alpha}\Big) \equiv \frac{\alpha}{1-\alpha} \; \frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha - 1} \quad (\lambda \geq 0) \tag{2.21}$$

2.2.1 We prove that f_{G_α}, as defined by (2.19), is a probability density, which is characterized by (2.10), or (2.21).

2.2.1.a) Let :

$$F_\alpha(\lambda) := \frac{\alpha}{1-\alpha} \; \frac{\lambda^{\alpha-1}-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-\lambda^\alpha} \tag{2.22}$$

Since the function f_{G_α} is continuous and integrable on $[0,1]$, in order to prove (2.19), we may use the inversion formula for the Stieltjes transform. Recall $\big($cf. [Wid], p. 340$\big)$ that if f is integrable and continuous, and if $S(f)$ denotes its Stieltjes transform :

$$S(f)(\lambda) = \int_0^\infty \frac{f(u)du}{\lambda+u} \tag{2.23}$$

we have :

$$f(u) = \lim_{\eta\to 0_+} \frac{Sf(-u-i\eta)-Sf(-u+i\eta)}{2i\pi} \tag{2.24}$$

Thus, to prove (2.19) amounts, thanks to the injectivity of the Stieltjes transform, to showing that :

$$\lim_{\eta\downarrow 0_+} \frac{F_\alpha(-u-i\eta)-F_\alpha(-u+i\eta)}{2i\pi} = \begin{cases} 0 \text{ if } u>1 \\ f_{G_\alpha}(u) \text{ if } u\in[0,1] \end{cases} \tag{2.25}$$

Formula (2.25) follows from an elementary computation ; in fact, we shall prove this result later in a more general framework $\big($cf **5.1.1** below$\big)$.

2.2.1.b) We prove that f_{G_α} is a probability density.

Since $f_{G_\alpha} \geq 0$, it suffices to show that : $\displaystyle\int_0^1 f_{G_\alpha}(u)\,du = 1$. Now, from (2.20), we obtain :

$$\begin{aligned} \int_0^1 f_{G_\alpha}(u)\,du &= \lim_{\lambda\to\infty} \lambda\, S(f_{G_\alpha})(\lambda) \\ &= \lim_{\lambda\to\infty} \frac{\alpha}{1-\alpha}\, \lambda \cdot \frac{\lambda^{\alpha-1}-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-\lambda^\alpha} \\ &= \lim_{\lambda\to\infty} \frac{\alpha}{1-\alpha} \, \frac{1-\Big(1+\frac{1}{\lambda}\Big)^{\alpha-1}}{\Big(1+\frac{1}{\lambda}\Big)^\alpha - 1} = 1. \end{aligned}$$

We also note that the equivalence between (2.20) and (2.21) follows from :

$$\begin{aligned} E(e^{-\lambda \mathfrak{e} G_\alpha}) &= E\Big(\frac{1}{1+\lambda G_\alpha}\Big) = \frac{1}{\lambda}\, E\Big[\frac{1}{\frac{1}{\lambda}+G_\alpha}\Big] = \frac{1}{\lambda}\, S(f_{G_\alpha})\Big(\frac{1}{\lambda}\Big) \\ &= \frac{1}{\lambda}\,\frac{\alpha}{1-\alpha}\,\frac{\left(\frac{1}{\lambda}\right)^{\alpha-1} - \left(\frac{1+\lambda}{\lambda}\right)^{\alpha-1}}{\left(\frac{1+\lambda}{\lambda}\right)^{\alpha} - \left(\frac{1}{\lambda}\right)^{\alpha}} = \frac{\alpha}{1-\alpha}\,\frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha - 1} \end{aligned} \tag{2.26}$$

2.2.2 We prove (2.18).

With the help of (2.21), and taking logarithmic derivatives on both sides of (2.18), the question amounts to showing : $\frac{\partial}{\partial\lambda}$ log $\big((1+\lambda)^\alpha - \lambda^\alpha\big) = -(1-\alpha)\int_0^\infty e^{-\lambda x} E(e^{-xG_\alpha})dx$ by (2.3), or :

$$\begin{aligned} \alpha\,\frac{(1+\lambda)^{\alpha-1}-\lambda^{\alpha-1}}{(1+\lambda)^\alpha - \lambda^\alpha} &= -(1-\alpha)\int_0^\infty e^{-\lambda x}dx \int_0^1 e^{-xu} f_{G_\alpha}(u)du \\ &= -(1-\alpha)\int_0^1 \frac{1}{\lambda+u}\, f_{G_\alpha}(u)du \quad \text{(Fubini)} \\ &= -(1-\alpha)\, E\Big[\frac{1}{\lambda+G_\alpha}\Big] \end{aligned} \tag{2.27}$$

However (2.27) is nothing else but (2.20). ■

The careful reader may have been surprised by the above proof, in particular by the proof given in **2.2.1.a)**, which may seem quite unnatural. Clearly, it is not in this manner that we discovered formula (2.18). Here is our original proof, which is more intuitive, but which, unfortunately, contains some non-rigorous features.

2.2.3 Another proof of formula (2.18)

2.2.3.a) Our aim is to find, from **2.2.1**, a r.v. G_α, taking values in $[0,1]$, such that :

$$E\Big[\frac{1}{1+\lambda G_\alpha}\Big] = \frac{\alpha}{1-\alpha}\,\frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha - 1} \quad (\lambda \geq 0) \tag{2.28}$$

When $\alpha = 1/2$, choosing for $G_{1/2}$ a r.v. with distribution beta$\Big(\frac{1}{2},\frac{1}{2}\Big)$, then the relation (2.28) is satisfied since, from the beta-gamma algebra :

$$\mathfrak{e}\cdot\beta_{(\alpha,1-\alpha)} \overset{\text{(law)}}{=} \gamma_{(\alpha)} \tag{2.29}$$

we deduce :

$$E[e^{-\lambda\mathfrak{e}\beta_{(\alpha,1-\alpha)}}] = E\Big(\frac{1}{1+\lambda\beta_{(\alpha,1-\alpha)}}\Big) = E(e^{-\lambda\gamma_{(\alpha)}}) = \frac{1}{(1+\lambda)^\alpha} \tag{2.30}$$

hence, for $\alpha = 1/2$, with : $G_{1/2} \overset{\text{(law)}}{=} \beta_{\left(\frac{1}{2},\frac{1}{2}\right)}$:

$$E\Big[\frac{1}{1+\lambda G_{1/2}}\Big] = \frac{1}{\sqrt{1+\lambda}} = \frac{1/2}{1-1/2}\,\frac{1-(1+\lambda)^{1/2}}{(1+\lambda)^{1/2}-1} \tag{2.31}$$

This particular result for $\alpha = 1/2$ invites to look whether for the density f_α of the r.v. G_α may be written in the form :

$$f_\alpha(u) = \int h_\gamma(u)\ \mu_\alpha(d\gamma)\ , \quad u \in [0,1] \tag{2.32}$$

where h_γ denotes here the density of a beta$(\gamma, 1-\gamma)$ variable, and $\mu_\alpha(d\gamma)$ a certain ≥ 0 measure. Since, from (2.30), one has :

$$\int_0^1 \frac{1}{1+\lambda u} h_\gamma(u) du = E\Big[\frac{1}{1+\lambda\beta_{(\gamma,1-\gamma)}}\Big] = \frac{1}{(1+\lambda)^\gamma} \tag{2.33}$$

the problem amounts to finding a measure $\mu_\alpha(d\gamma)$ such that :

$$\int_0^1 \frac{f_\alpha(u)du}{1+\lambda u} = \int \frac{1}{(1+\lambda)^\gamma}\mu_\alpha(d\gamma) = \frac{\alpha}{1-\alpha}\ \frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha - 1} \quad (\lambda \geq 0) \tag{2.34}$$

2.2.3.b) Searching for $\mu_\alpha(d\gamma)$ such that (2.34) is satisfied.
We replace in (2.34) $(1+\lambda)$ by e^t $(t \geq 0)$, and we obtain :

$$\frac{\alpha}{1-\alpha}\ \frac{1-e^{t(\alpha-1)}}{e^{t\alpha}-1} = \frac{\alpha}{1-\alpha}\Big(\sum_{m=0}^\infty e^{-t(m+1)\alpha} - \sum_{n=0}^\infty e^{-t(n\alpha+1)}\Big) = \int e^{-\alpha t}\mu_\alpha(d\gamma) \tag{2.35}$$

Consequently, since both sides of (2.35) are Laplace transforms, we obtain :

$$\mu_\alpha(d\gamma) = \frac{\alpha}{1-\alpha}\Big\{\sum_{m=0}^\infty \delta_{(m+1)\alpha}(d\gamma) - \sum_{n=0}^\infty \delta_{(n\alpha+1)}(d\gamma)\Big\} \tag{2.36}$$

We shall now discuss two cases :

i) $\alpha = \frac{1}{p}$, p an integer, $p \geq 2$.

In this case, the following computation is entirely rigorous. In formula (2.36), one finds only $(p-1)$ terms, since : $(p-1+1)\alpha = p \cdot \frac{1}{p} = 1 = 0 \cdot \alpha + 1$. Hence :

$$\mu_\alpha(d\gamma) = \frac{\alpha}{1-\alpha}\sum_{k=1}^{p-1}\delta_{k\alpha}(d\gamma) = \frac{1}{p-1}\sum_{k=1}^{p-1}\delta_{\frac{k}{p}}(d\gamma) \tag{2.37}$$

so that, plugging this value of μ_α in (2.32), we obtain :

$$\begin{aligned} f_\alpha(u) &= \frac{1}{p-1}\sum_{k=1}^{p-1} h_{\frac{k}{p}}(u) = \frac{1}{p-1}\sum_{k=1}^{p-1}\frac{u^{\frac{k}{p}-1}(1-u)^{-\frac{k}{p}}}{\Gamma\Big(\frac{k}{p}\Big)\Gamma\Big(1-\frac{k}{p}\Big)} \\ &= \frac{1}{p-1}\sum_{k=1}^{p-1}\frac{\sin\big(\pi\frac{k}{p}\big)}{\pi}\ u^{\frac{k}{p}-1}(1-u)^{-\frac{k}{p}} \end{aligned}$$

from the formula of complements for the gamma function : $\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$ (cf [Leb], p. 3). Hence :

$$\begin{aligned}
f_\alpha(u) &= \frac{1}{(p-1)\pi u}\,\mathrm{Im}\left(\sum_{k=1}^{p-1} e^{\frac{i\pi k}{p}}\left(\frac{u}{1-u}\right)^{\frac{k}{p}}\right)\\
&= \frac{1}{(p-1)\pi u}\,\mathrm{Im}\left(\frac{e^{i\frac{\pi}{p}}\left(\frac{u}{1-u}\right)^{\frac{1}{p}}+\frac{u}{1-u}}{1-e^{i\frac{\pi}{p}}\left(\frac{u}{1-u}\right)^{\frac{1}{p}}}\right)\\
&= \frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi}\,\frac{u^{\alpha-1}(1-u)^{\alpha-1}}{(1-u)^{2\alpha}-2(1-u)^\alpha u^\alpha\cos(\pi\alpha)+u^{2\alpha}},\ \text{with } \alpha=\frac{1}{p}\cdot
\end{aligned}$$

ii) <u>α is not of the form $\frac{1}{p}$, with p an integer, $p\geq 2$.</u>

Plugging (2.36) in (2.32), we get :

$$f_\alpha(u) = \frac{\alpha}{1-\alpha}\left(\sum_{m=0}^{\infty}\frac{u^{(m+1)\alpha-1}(1-u)^{-(m+1)\alpha}}{\Gamma\big((m+1)\alpha\big)\Gamma\big(1-(m+1)\alpha\big)}-\sum_{n=0}^{\infty}\frac{u^{n\alpha}(1-u)^{-n\alpha-1}}{\Gamma(n\alpha+1)\Gamma(-n\alpha)}\right) \tag{2.38}$$

$$= \frac{\alpha}{1-\alpha}\left\{\frac{1}{u\pi}\sum_{m=0}^{\infty}\sin\big(\pi\alpha(m+1)\big)\left(\frac{u}{1-u}\right)^{(m+1)\alpha} - \frac{1}{(1-u)\pi}\sum_{n=0}^{\infty}\sin(\pi\alpha n)\left(\frac{u}{1-u}\right)^{n\alpha}\right\} \tag{2.39}$$

again from the formula of complements. Hence :

$$\begin{aligned}
f_\alpha(u) &= \frac{\alpha}{1-\alpha}\left\{\frac{1}{u\pi}\,\mathrm{Im}\left(\frac{e^{i\pi\alpha}\left(\frac{u}{1-u}\right)^\alpha}{1-e^{i\pi\alpha}\left(\frac{u}{1-u}\right)^\alpha}\right)+\frac{1}{(1-u)\pi}\,\mathrm{Im}\left(\frac{1}{1-e^{i\pi\alpha}\left(\frac{u}{1-u}\right)^\alpha}\right)\right\}\\
&= \frac{\alpha}{1-\alpha}\,\frac{1}{\pi}\left\{\frac{1}{(1-u)^{2\alpha}-2(1-u)^\alpha u^\alpha\cos(\pi\alpha)+u^{2\alpha}}\right.\\
&\qquad\left.\cdot\big(\sin(\pi\alpha)\big)u^{\alpha-1}(1-u)^{\alpha-1}(1-u+u)\right\}\\
&= \frac{\alpha}{1-\alpha}\,\frac{\sin(\pi\alpha)}{\pi}\,\frac{u^{\alpha-1}(1-u)^{\alpha-1}}{(1-u)^{2\alpha}-2(1-u)^\alpha u^\alpha\cos(\pi\alpha)+u^{2\alpha}}\quad,\ u\in[0,1].
\end{aligned}$$

In fact, this computation may be made quite rigorous with the help of the two following arguments :

- Although the function $h_\gamma(u)$ is a density only for $\gamma\in[0,1]$, we may replace everywhere in this computation h_γ by its holomorphic prolongation (with respect to the γ variable).

- The two series which appear in this computation may be "reduced" to :

$$\sum_{n=0}^{\infty} e^{i\pi n\alpha}\Big(\frac{u}{1-u}\Big)^{n\alpha}$$

which only converges for $\frac{u}{1-u} < 1$, i.e. : for $u < \frac{1}{2}\cdot$ But, it is not difficult to see that the density f_α, which we are trying to obtain, is such that $f_\alpha(u) = f_\alpha(1-u) \quad (u \in [0,1])$ (e.g. see (1.19) and point 3 of Theorem 2). Thus, it suffices to consider $u \in [0, 1/2]$, and it is precisely for these values of u that the previous series converges.

2.2.4 We prove that Δ_α is self-decomposable.
From Lukacs $\big($[Luk], p. 164$\big)$, this is equivalent to the property that $x \to x\nu_\alpha(x)$ is a decreasing function of x, where ν_α denotes the density of the Lévy measure of Δ_α. This is satisfied, since :

$$\nu_\alpha(x) = \frac{(1-\alpha)}{x}\, E\big[\exp(-xG_\alpha)\big].$$

In fact, all generalized gamma convolutions are self-decomposable.

2.2.5 Remark 2.2 :
It is well-known that a self-decomposable distribution σ is the invariant measure of a generalized Ornstein-Uhlenbeck process $(Y_t,\, t \geq 0)$, i.e. : a process which solves :

$$dY_t = -Y_t\, dt + dZ_t \tag{2.40}$$

where $(Z_t,\, t \geq 0)$ is a Lévy process $\big($cf [Sat] ; [Sch], p. 49$\big)$. Furthermore, if Φ_Z (resp. Φ_σ) denotes the characteristic exponent of Z (resp. σ), one has :

$$\Phi_Z(\lambda) = \lambda \frac{\Phi'_\sigma(\lambda)}{\Phi_\sigma(\lambda)} \quad (\lambda \geq 0) \tag{2.41}$$

We deduce from this formula that : if w (resp. u) denotes the density of the Lévy measure of Z (resp. σ) then :

$$w(x) = -u(x) - xu'(x) \tag{2.42}$$

We apply this in the case where σ_α is the law of Δ_α, that is, from (1.14) :

$$\sigma_\alpha(dx) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-\alpha-1}(1-e^{-x})\, 1_{[0,\infty[}(x)dx$$

Then, there exists a Lévy process $(Z_t^{(\alpha)},\, t \geq 0)$ with Lévy exponent Φ_α and Lévy density w_α such that the process $(Y_t^{(\alpha)},\, t \geq 0)$, which solves :

$$dY_t^{(\alpha)} = -Y_t^{(\alpha)} dt + d\, Z_t^{(\alpha)} \tag{2.43}$$

admits σ_α as its invariant probability measure. Formulae (2.41) and (2.42) now become :

$$\Phi_\alpha(\lambda) = \alpha\lambda \frac{\lambda^{\alpha-1} - (1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha - \lambda^\alpha},\ w_\alpha(x) = (1-\alpha)E\big[G_\alpha\big(\exp(-xG_\alpha)\big)\big] \tag{2.44}$$

2.3 Proof of part 3) of Theorem 1.1

Let $K_\alpha \overset{\text{(law)}}{=} \mathfrak{e}/G_\alpha$ *with* $\mathfrak{e}$ *and* G_α *independent. Then :*

i) There exists a $\big((1-\alpha),\, K_\alpha\big)$ *positive compound Poisson process* $(Y_t, t \geq 0)$ *such that :*

$$\Delta_\alpha \overset{\text{(law)}}{=} \int_0^\infty e^{-t} dY_t \tag{2.45}$$

ii) Δ_α *satisfies the following affine equation :*

$$\Delta_\alpha \overset{\text{(law)}}{=} U^{1/1-\alpha}(\Delta_\alpha + K_\alpha) \tag{2.46}$$

where, on the RHS of (2.46), U, Δ_α *and* K_α *are independent, and* U *is uniformly distributed on* $[0,1]$.

2.3.1 Proof of (2.45) and (2.46) :

It hinges upon the following proposition.

Proposition 2.3. *Let* $(Y_t, t \geq 0)$ *denote a subordinator, without drift, and with Lévy measure* μ*. Let :*

$$X := \int_0^\infty e^{-t} dY_t \tag{2.47}$$

We assume that $X < \infty$ *a.s. which, from Jurek-Vervaat ([J,V]), see also Erikson-Maller ([E,M]), is equivalent to :*

$$\int_{[1,\infty[} (\log x)\mu(dx) < \infty \tag{2.48}$$

Then :

$$i) \quad E(e^{-\lambda X}) = \exp\Big(-\int_0^\infty (1-e^{-\lambda v})\mu\big([v,\infty]\big)\frac{dv}{v} \tag{2.49}$$

In particular, X *is self-decomposable.*

ii) If, morever, $(Y_t, t \geq 0)$ *is a* (γ, K) *compound Poisson process, (i.e. :* $\gamma := \mu(\mathbb{R}_+) < \infty$*), then :*

$$X \overset{\text{(law)}}{=} U^{1/\gamma}(X + K) \tag{2.50}$$

where, on the RHS of (2.50), U, X *and* K *are independent, and* U *is uniform on* $[0,1]$.

2.3.2 We prove that Proposition 2.3. implies (2.45) and (2.46)
We know, from (1.16), that :

$$E(e^{-\lambda\Delta_\alpha}) = \exp\Big\{ -(1-\alpha)\int_0^\infty (1-e^{-\lambda x})E(e^{-xG_\alpha})\,\frac{dx}{x}\Big\} \tag{2.51}$$

On the other hand, from the definition of K_α, we have :

$$P(K_\alpha \geq x) = P\Big(\frac{\mathfrak{e}}{G_\alpha} \geq x\Big) = P(\mathfrak{e} > xG_\alpha) = E(e^{-xG_\alpha}) \tag{2.52}$$

We denote by μ_α the law of K_α. Then, replacing $E\big(\exp(-xG_\alpha)\big)$ in (2.51) by its value as obtained in (2.52), we get :

$$E(e^{-\lambda\Delta_\alpha}) = \exp\Big\{ -(1-\alpha)\int_0^\infty (1-e^{-\lambda x})\mu_\alpha\big([x,\infty[\big)\,\frac{dx}{x}\,. \tag{2.53}$$

It then suffices to compare (2.51) and (2.49), then we apply Proposition 2.3 to obtain (2.45) and (2.46), with $\gamma = 1-\alpha$.

2.3.3 Proof of Proposition 2.3. (but, see also [J,V] for the original proof)
2.3.3.a) Approximating $X = \displaystyle\int_0^\infty e^{-t}dY_t$ by the Riemann sums $\displaystyle\sum_i e^{-t_i}(Y_{t_{i+1}} - Y_{t_i})$ we obtain :

$$\begin{aligned} E(e^{-\lambda X}) &= \exp\Big\{ -\int_0^\infty dt \int_0^\infty (1-e^{-\lambda e^{-t}x})\mu(dx)\Big\} \\ &= \exp\Big\{ -\int_0^\infty \mu(dx)\Big(\int_0^x (1-e^{-\lambda v})\frac{dv}{v}\Big)\Big\} \end{aligned} \tag{2.54}$$

(after making the change of variables $e^{-t}x = v$).

$$= \exp\Big\{ -\int_0^\infty (1-e^{-\lambda v})\mu\big([v,\infty]\big)\,\frac{dv}{v}\Big\} \quad \text{by Fubini's Theorem.}$$

2.3.3.b) We prove point *ii)* of Proposition 2.3.
Recall that $(Y_t, t \geq 0)$ may be represented as :

$$Y_t = \sum_{i=1}^{N_t} K_i$$

where $(N_t, t \geq 0)$ denotes a Poisson process with parameter γ, independent from the sequence of i.i.d. variables (K_i). Let T_1 be the first jump time of $(N_t, t \geq 0)$. Then, one has :

$$\begin{aligned} X &= \int_0^\infty e^{-t}dY_t = \int_0^{T_1} e^{-t}dY_t + \int_{T_1}^\infty e^{-t}dY_t \\ &= e^{-T_1}K_1 + e^{-T_1}\widetilde{X} \end{aligned}$$

where $\widetilde{X}$ is independent from (T_1, K_1), and is distributed as X. This proves (2.46), since, as T_1 is exponentially distributed, with parameter γ, one has :

$$e^{-T_1} \stackrel{\text{(law)}}{=} U^{1/\gamma}$$

2.3.3.c) Another proof of (2.50).
If we denote by θ, resp. φ, the Laplace transform of X, resp. K, then, the relation (2.50) is equivalent to :

$$\begin{aligned}
\theta(\lambda) &= E\Big(\int_0^1 e^{-\lambda u^{1/\gamma}(X+K)}du\Big) = \gamma E\Big(\int_0^1 e^{-\lambda v(X+K)}v^{\gamma-1}dv\Big) \\
&= \frac{\gamma}{\lambda^\gamma}\int_0^\lambda \theta(u)\varphi(u)u^{\gamma-1}du, \quad \text{i.e. :} \\
\lambda^\gamma\theta(\lambda) &= \gamma\int_0^\lambda \theta(u)\varphi(u)u^{\gamma-1}du \qquad (2.55)
\end{aligned}$$

which, taking derivatives, is equivalent to :

$$\begin{aligned}
-\frac{\theta'(\lambda)}{\theta(\lambda)} &= \gamma\big(1-\varphi(\lambda)\big) \quad (\lambda \geq 0), \quad \text{hence :} \\
\theta(\lambda) &= E(e^{-\lambda X}) = \exp\Big\{ -\int_0^\infty (1-e^{-\lambda v})\gamma\mu_K\big([v,\infty[\big)\,\frac{dv}{v}\Big\} \qquad (2.56)
\end{aligned}$$

where μ_K denotes the law of K. It now remains to observe that the Lévy measure of subordinator $(Y_t, t \geq 0)$ is equal to $\gamma \cdot \mu_K$ and then to compare (2.54) and (2.56).

2.4 Remark 2.4

We come back to the result of M. Winkel (cf paragraph 1.2). Let $(\tau_l, l \geq 0)$ be a subordinator, without drift and with Lévy exponent Φ. Let :

$$\Delta^{(\tau)} := \Delta_{\mathfrak{e}}$$

with the notations of paragraph 1.2. Hence, by (1.9) :

$$E(e^{-\lambda\Delta^{(\tau)}}) = \frac{\Phi(1+\lambda)-\Phi(\lambda)}{\Phi(1)} \qquad (2.57)$$

A natural question is the following : which are the positive r.v.'s Δ such that :

$$\Delta \overset{(\text{law})}{=} \Delta^{(\tau)}$$

for some subordinator $(\tau_l, l \geq 0)$? The answer to this question is elementary ; for any positive r.v. Δ there exists a unique subordinator $(\tau_l, l \geq 0)$ without drift and with Lévy exponent Φ, with $\Phi(1) = 1$, such that :

$$\Delta \overset{(\text{law})}{=} \Delta^{(\tau)} \qquad (2.58)$$

2.4.1 Proof of Remark 2.4

Let ψ the Laplace transform of Δ and denote by μ_Δ the law of Δ. Then :

$$\psi(\lambda) = E(e^{-\lambda\Delta}) = \int_0^\infty e^{-\lambda x}d\mu_\Delta(x) = \int_0^\infty e^{-\lambda x}(1-e^{-x})\frac{\mu_\Delta(dx)}{(1-e^{-x})}$$

Let $\widetilde{\nu}_\Delta$ be defined by :

$$\widetilde{\nu}_\Delta(dx) = \frac{1}{(1-e^{-x})}\ \mu_\Delta(dx) \tag{2.59}$$

There is no difficulty in showing that $\widetilde{\nu}_\Delta$ is a Lévy measure, i.e :

$$\int_0^\infty (x \wedge 1)\widetilde{\nu}_\Delta(dx) < \infty$$

Let Φ denote the associated Bernstein function and $(\tau_l, l \geq 0)$ the corresponding subordinator

$$\Phi(\lambda) = \int_0^\infty (1-e^{-\lambda x})\widetilde{\nu}_\Delta(dx)$$

We have
$$\begin{aligned}\psi(\lambda) &= \int_0^\infty e^{-\lambda x}(1-e^{-x})\widetilde{\nu}_\Delta(dx) \\ &= \int_0^\infty (1-e^{-(\lambda+1)x})\widetilde{\nu}_\Delta(dx) - \int_0^\infty (1-e^{-\lambda x})\widetilde{\nu}_\Delta(dx) \\ &= \Psi(1+\lambda) - \Psi(\lambda) \end{aligned} \tag{2.60}$$

It is clear that $\Psi(0) = 1 = \Phi(1) - \Phi(0) = \Phi(1)$. Then, from (2.57) and (2.60), we obtain :

$$E(e^{-\lambda\Delta^{(\tau)}}) = \Phi(1+\lambda) - \Phi(\lambda) = \Psi(\lambda) = E(e^{-\lambda\Delta})$$

that is

$$\Delta^{(\tau)} \overset{\text{(law)}}{=} \Delta$$

The uniqueness of $(\tau_l, l \geq 0)$ may be proven using similar arguments.

3 Properties of the variables G_α $(0 < \alpha < 1)$. Proofs of Theorems 1.2 and 1.3.

We begin with point 1) of Theorem 1.2.

3.1 *$G_{1/2}$ is arc sine distributed; i.e. it is distributed as beta $\left(\frac{1}{2}, \frac{1}{2}\right)$:*

$$f_{G_{1/2}}(u) = \frac{1}{\pi}\ \frac{1}{\sqrt{u(1-u}}\ 1_{[0,1]}(u) \tag{3.1}$$

Proof of (3.1) :
It suffices to take $\alpha = 1/2$ in (1.17), or to note that :

$$\begin{aligned} E\left(e^{-\lambda\mathfrak{e}\beta_{\left(\frac{1}{2},\frac{1}{2}\right)}}\right) &= E(e^{-\lambda\gamma_{1/2}}) = \frac{1}{\sqrt{1+\lambda}} = \frac{1-(1+\lambda)^{-1/2}}{(1+\lambda)^{1/2}-1} \\ &= E\left(\frac{1}{1+\lambda\beta_{\left(\frac{1}{2},\frac{1}{2}\right)}}\right) = E\left(\frac{1}{1+\lambda G_{1/2}}\right) \quad \text{from(1.19)} \end{aligned}$$

3.2 Proof of point 2) of Theorem 1.2

If $\alpha = 1/p$, with p an integer, $p \geq 2$, then :

$$f_{G_\alpha}(u) = f_{G_{1/p}}(u) = \frac{1}{\pi(p-1)} \sum_{i=1}^{p-1} \sin\left(\frac{\pi i}{p}\right) u^{\frac{i}{p}-1}(1-u)^{-\frac{i}{p}} 1_{[0,1]}(u)$$

In fact, this is the formula following (2.37), which was proven above, in **2.2.3.b)**.
We obtain (1.26) from (1.25) after the change of index $j = p - i$.

3.3 Proof of point 3) of Theorem 1.2

$$G_\alpha \overset{\text{(law)}}{=} 1 - G_\alpha \tag{3.2}$$

Thanks to (1.17), or (1.18), this relation is obvious

3.4 Proof of point 4) of Theorem 1.2 :

G_α converges in law as $\alpha \to 1$, to a uniformly distributed r.v. $[0,1]$.

It is sufficient, to prove this assertion, to observe that :

$$\lim_{\alpha\to 1} E\Big[\frac{1}{1+\lambda G_\alpha}\Big] = \lim_{\alpha \to 1} \frac{\alpha}{1-\alpha}\,\frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha - 1} = \frac{\log(1+\lambda)}{\lambda} \quad \text{from (1.19)}$$

and, if U is a uniform r.v. on $[0,1]$:

$$E\Big[\frac{1}{1+\lambda U}\Big] = \int_0^1 \frac{1}{1+\lambda u}\,du = \frac{1}{\lambda}\big[\log(1+\lambda) - \log(1)\big] = \frac{\log(1+\lambda)}{\lambda}$$

3.5 Proof of point 5) of Theorem 1.2

G_α converges in law, as $\alpha \to 0$, to a r.v. G_0 which satisfies :

$$i) \quad f_{G_0}(u) = \frac{1}{\pi}\Big(\int_0^1 \big(\sin(\pi\beta)\big)u^{\beta-1}(1-u)^{-\beta}d\beta\Big) \bullet 1_{[0,1]}(u) \tag{3.3}$$

$$= \frac{1}{u(1-u)}\,\frac{1}{\pi^2 + \big(\log\frac{1-u}{u}\big)^2} \tag{3.4}$$

$$ii) \quad G_0 \overset{\text{(law)}}{=} \frac{1}{1+\exp \pi C}, \quad \textit{where } C \textit{ is a standard Cauchy variable} \tag{3.5}$$

3.5.1 Proof of (3.3) and (3.4).

3.5.1.a) We first note that formula (3.3) indicates, with the notations in formula (2.32) that the measure $\nu_0(d\gamma)$ is Lebesgue measure on the interval $[0,1]$. On the other hand, from (1.25) :

$$f_{G_{\frac{1}{p}}}(u) = \Big(\frac{1}{\pi(p-1)} \sum_{i=1}^{p-1} \sin\left(\frac{\pi i}{p}\right) u^{\frac{i}{p}-1}(1-u)^{-\frac{i}{p}}\Big) \bullet 1_{[0,1]}(u)$$

$$\underset{p\to\infty}{\longrightarrow} \frac{1}{\pi}\Big(\int_0^1 \big(\sin(\pi\beta)\big)u^{\beta-1}(1-u)^{-\beta}d\beta\Big) \bullet 1_{[0,1]}(u)$$

which proves (3.3). In fact, we have only studied the limit, as $p \to \infty$, of $f_{G_{\frac{1}{p}}}$. But, the explicit formula (1.17) which gives f_{G_α}, easily shows that as $\alpha \downarrow 0$, f_{G_α} converges (to f_{G_0}).

3.5.1.b) The relation (3.4) follows from :

$$\begin{aligned}\frac{1}{\pi}\int_0^1 \big(\sin(\pi\beta)\big)u^{\beta-1}(1-u)^{-\beta}d\beta &= \frac{1}{u\pi}\mathrm{Im}\int_0^1 e^{\beta\left(i\pi+\log\frac{u}{1-u}\right)}d\beta \\ &= \frac{1}{u(1-u)}\,\frac{1}{\pi^2+\left(\log\frac{u}{1-u}\right)^2}\end{aligned}$$

3.5.1.c) We now prove (3.5) :
We already observe that, from (3.4) :

$$\begin{aligned}E\Big(\frac{1}{1+\lambda G_0}\Big) &= \int_0^1 \frac{1}{1+\lambda u}\,\frac{1}{u(1-u)}\,\frac{1}{\pi^2+\left(\log\frac{u}{1-u}\right)^2}\,du \\ &= \int_0^\infty \frac{1+v}{1+v+\lambda v}\,\frac{1}{\pi^2+\log^2 v}dv \quad \text{(after the change of variable } \frac{u}{1-u}=v) \\ &= \frac{1}{\pi}\int_{-\infty}^\infty \frac{1+e^{\pi w}}{1+(\lambda+1)e^{\pi w}}\,\frac{dw}{1+w^2} \quad \text{(after the change of variable } \log v=\pi w)\end{aligned}$$

whereas :

$$\begin{aligned}E\left(\frac{1}{1+\frac{\lambda}{1+e^{\pi C}}}\right) &= E\Big(\frac{1+e^{\pi C}}{1+\lambda+e^{\pi C}}\Big) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{1+e^{-\pi x}}{1+\lambda+e^{-\pi x}}\,\frac{dx}{1+x^2} \\ &= \frac{1}{\pi}\int_{-\infty}^\infty \frac{1+e^{\pi w}}{1+(1+\lambda)e^{\pi w}}\,\frac{dw}{1+w^2}\end{aligned}$$

which yields (3.4). Below (cf Remark 3.2.), we shall give another proof of the convergence in law of G_α, as $\alpha \to 0$, towards : $\dfrac{1}{1+\exp(\pi C)}$.
This ends the proof of Theorem 1.2. ∎

3.6 Remark 3.1. (A relation between G_0 and the gamma subordinator).

3.6.1 For any λ and μ positive reals, we write, using (1.15) and (1.16) :

$$\begin{aligned}&\frac{E\big(\exp\big(-(\lambda+\mu)\Delta_\alpha\big)\big)}{E\big(\exp(-\mu\Delta_\alpha)\big)} = \frac{(1+\lambda+\mu)^\alpha-(\lambda+\mu)^\alpha}{(1+\mu)^\alpha-\mu^\alpha} \\ &= \exp\Big\{-(1-\alpha)\int_0^\infty \big(1-e^{-(\lambda+\mu)x}-1+e^{-\mu x}\big)E(e^{-xG_\alpha})\frac{dx}{x}\Big\} \qquad (3.6)\end{aligned}$$

Letting $\alpha \to 0$ on both sides of (3.6), and using the already proven fact that $G_\alpha \underset{\alpha\to 0}{\overset{law}{\to}} G_0$, we obtain :

$$\frac{\log(1+\lambda+\mu)-\log(\lambda+\mu)}{\log(1+\mu)-\log\mu} = \exp\Big\{-\int_0^\infty e^{-\mu x}(1-e^{-\lambda x})E(e^{-xG_0})\,\frac{dx}{x}\Big\} \qquad (3.7)$$

3.6.2 We denote by Φ_μ the Lévy exponent of the subordinator $\Big(\frac{1}{\mu}\gamma_t, t \geq 0\Big)$, where $(\gamma_t, t \geq 0)$ denotes the standard gamma subordinator. Thus :

$$\begin{aligned} E(e^{-\frac{\lambda}{\mu}\gamma_t}) &= \frac{1}{\big(1+\frac{\lambda}{\mu}\big)^t} = \exp\Big\{-t\big(\log(\lambda+\mu)-\log\mu\big)\Big\} \\ \text{i.e.} \qquad & \Phi_\mu(\lambda) = \log(\lambda+\mu)-\log\mu \end{aligned} \tag{3.8}$$

Hence, formula (3.7) writes :

$$\frac{\Phi_\mu(1+\lambda)-\Phi_\mu(\lambda)}{\Phi_\mu(1)} = \exp\Big\{-\int_0^\infty (1-e^{-\lambda x})e^{-\mu x}E(e^{-xG_0})\,\frac{dx}{x}\Big\} \tag{3.9}$$

3.6.3 Let $(X_t^{(\mu)}, t \geq 0)$ denote a diffusion process whose inverse local time at 0, $(\tau_l^{(\mu)}, l \geq 0)$ is distributed as $\Big(\frac{1}{\mu}\,\gamma_l, l \geq 0\Big)$. Such a diffusion $(X_t^{(\mu)}, t \geq 0)$ has been described explicitly by C. Donati-Martin and M. Yor $\big($cf [D.M, Y]$\big)$, as an illustration of Krein's representation of subordinators. Furthermore, we define, for $t \geq 0$:

$$\begin{aligned} & g_t^{(\mu)} := \sup\{s \leq t, X_s^{(\mu)} = 0\}, \quad d_t^{(\mu)} := \inf\{s \geq t,\ X_s^{(\mu)} = 0\} \\ \text{and} \qquad & \Delta_{(\mu)} := d_{\mathfrak{e}}^{(\mu)} - g_{\mathfrak{e}}^{(\mu)} \end{aligned} \tag{3.10}$$

where $\mathfrak{e}$ denotes a standard exponential variable, independent from $(X_t^{(\mu)}, t \geq 0)$. Then, as we apply Lemma 2.1., formula (3.9) becomes :

$$E(e^{-\lambda\Delta_{(\mu)}}) = \exp\Big\{-\int_0^\infty (1-e^{-\lambda x}) \quad E(e^{-x(\mu+G_0)})\,\frac{dx}{x}\Big\} \tag{3.11}$$

It follows from (3.11) that $\Delta_{(\mu)}$ is self-decomposable.
We note that this formula (3.11) is quite similar to (1.16), when we replace :

- the stable (α) process $(\tau_l^{(\alpha)}, l \geq 0)$ by the gamma process $\Big(\frac{1}{\mu}\,\gamma_l, l \geq 0\Big)$;
- the r.v. G_α by the r.v. $\mu + G_0$ $\big($and also replace the coefficient $(1-\alpha)$ by 1 in (1.16)$\big)$.

It is tempting to let μ tend to 0 in (3.11). However, this is not possible, for two reasons :

i) the process $\Big(\frac{1}{\mu}\,\gamma_l, l \geq 0\Big)$ does not converge as $\mu \to 0$;

ii) the measure $\frac{1}{x}\,E(e^{-xG_0})\,dx$ is not integrable near ∞ $\Big($as $E\Big(\log\frac{1}{G_0}\Big) = \infty\Big)$ hence it does not define a Lévy measure.

3.7 Proof of Theorem 1.3 (Links between the r.v.'s G_α, the unilateral stable variables, and the Mittag-Leffler distribution).

We refer the reader to the Introduction, paragraph 5, for the definitions of T_μ, T'_μ, Z_μ and M_μ $\quad\big(\mu \in]0,1[\big)$.

3.7.1 Proof of point 1) of Theorem 1.3 :
Z_μ *admits the density :*

$$f_{Z_\mu}(x) = \frac{\sin(\pi\mu)}{\pi\mu} \, \frac{1}{x^2 + 2x\cos(\pi\mu) + 1} \, 1_{[0,\infty[}(x) \tag{3.12}$$

In fact, formula (3.12) is due to Lamperti $\big($[Lamp]$\big)$. A proof of (3.12) is also found in Chaumont-Yor $\big($cf [CY], ex. 4.21, p. 116$\big)$. We refer the interested reader to this proof.

3.7.2 Proof of point 2.i) of Theorem 1.3 :

$$G_\alpha \overset{\text{(law)}}{=} \frac{(Z_{1-\alpha})^{\frac{1}{\alpha}}}{1 + (Z_{1-\alpha})^{\frac{1}{\alpha}}} \overset{\text{(law)}}{=} \frac{(T_{1-\alpha})^{\frac{1-\alpha}{\alpha}}}{(T'_{1-\alpha})^{\frac{1-\alpha}{\alpha}} + (T_{1-\alpha})^{\frac{1-\alpha}{\alpha}}} \tag{3.13}$$

We now prove (3.13).
For this purpose, we shall show that $\left(\frac{G_\alpha}{1-G_\alpha}\right)^\alpha$ is distributed as $Z_{1-\alpha}$, which implies (3.13).
Indeed, for any $h : \mathbb{R}_+ \to \mathbb{R}_+$, Borel, one has :

$$\begin{aligned}
& E\left[h\left(\left(\frac{G_\alpha}{1-G_\alpha}\right)^\alpha\right)\right] \\
= \; & \frac{\alpha}{1-\alpha} \, \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty h\left(\left(\frac{u}{1-u}\right)^\alpha\right) \frac{u^{\alpha-1}(1-u)^{\alpha-1}}{(1-u)^{2\alpha} - 2(1-u)^\alpha u^\alpha \cos(\pi\alpha) + u^{2\alpha}} \, du \\
= \; & \frac{\sin(\pi\alpha)}{\pi(1-\alpha)} \int_0^\infty h(x) \, \frac{dx}{x^2 - 2x\cos(\pi\alpha) + 1}
\end{aligned}$$

$\left(\text{after making the change of variables : } \left(\frac{u}{1-u}\right)^\alpha = x\right)$

$$\begin{aligned}
= \; & \frac{\sin\big(\pi(1-\alpha)\big)}{\pi(1-\alpha)} \int_0^\infty h(x) \, \frac{dx}{x^2 + 2x\cos\big(\pi(1-\alpha)\big) + 1} \\
= \; & E\big[h(Z_{1-\alpha})\big] \quad \text{from (3.12)}
\end{aligned}$$

3.7.3 Proof of point 2. ii) of Theorem 1.3 :

$$G_\alpha \overset{\text{(law)}}{=} \frac{(M_{1-\alpha})^{\frac{1}{\alpha}}}{(M_{1-\alpha})^{\frac{1}{\alpha}} + (M'_{1-\alpha})^{\frac{1}{\alpha}}} \tag{3.14}$$

where, on the RHS of (3.14), $M_{1-\alpha}$ and $M'_{1-\alpha}$ denote two independent r.v.'s, with the Mittag-Leffler distribution with parameter $(1-\alpha)$.
To prove (3.14), we use (cf, Introduction, paragraph 5)

$$E(M_\mu^n) = \frac{\Gamma(n+1)}{\Gamma(\mu n + 1)} \qquad (n > -1) \tag{3.15}$$

On the other hand, using the elementary formula :

$$
\begin{aligned}
E\Big[\frac{1}{T_\mu^{\mu n}}\Big] &= \frac{1}{\Gamma(\mu n)}\int_0^\infty u^{\mu n-1}E(e^{-uT_\mu})du \\
&= \frac{1}{\Gamma(\mu n)}\int_0^\infty u^{\mu n-1}e^{-u^\mu}du \\
&= \frac{\Gamma(1+n)}{\Gamma(\mu n+1)} \quad (n>-1) \qquad (3.16)
\end{aligned}
$$

Now, comparing (3.16) et (3.15), we deduce that :

$$M_\mu \overset{\text{(law)}}{=} \frac{1}{(T_\mu)^\mu},$$

and (3.14) now follows from (3.13).

3.8 Remark 3.2.

We present here another proof of the convergence of G_α, as $\alpha \to 0$, to $\dfrac{1}{1+e^{\pi C}}$ where C is a standard Cauchy r.v. It suffices to prove that :

$$\log(1-G_\alpha)-\log(G_\alpha) \underset{\alpha\to 0}{\overset{\text{law}}{\to}} \pi C$$

or, by (3.13), that :

$$\frac{1}{\alpha}\big(\log(T'_{1-\alpha})-\log(T_{1-\alpha})\big) \underset{\alpha\to 0}{\overset{\text{law}}{\longrightarrow}} \pi C \qquad (3.17)$$

where $T_{1-\alpha}$ and $T'_{1-\alpha}$ are two independent copies of a one-sided stable $(1-\alpha)$ r.v. But : $T_{1-\alpha} \underset{\alpha\to 0}{\longrightarrow} 1$ in probability. Hence (3.17) is equivalent to :

$$\frac{1}{\alpha}(T'_{1-\alpha}-T_{1-\alpha}) \underset{\alpha\to 0}{\overset{\text{law}}{\longrightarrow}} \pi C \qquad (3.18)$$

We prove (3.18) :

$$
\begin{aligned}
E(e^{i\frac{\lambda}{\alpha}T_{1-\alpha}}) &= E\Big(\exp\Big(\frac{\lambda}{\alpha}e^{i\frac{\pi}{2}}T_{1-\alpha}\Big)\Big) = \exp\Big(-\frac{|\lambda|^{1-\alpha}}{\alpha^{1-\alpha}}e^{i\frac{\pi}{2}(1-\alpha)}\Big) \\
&= \exp\Big\{-\frac{|\lambda|^{1-\alpha}}{\alpha^{1-\alpha}}\Big(\cos\Big(\frac{\pi}{2}(1-\alpha)\Big)+i\sin\Big(\frac{\pi}{2}(1-\alpha)\Big)\Big)\Big\}
\end{aligned}
$$

Hence :

$$
\begin{aligned}
E(e^{i\frac{\lambda}{\alpha}(T_{1-\alpha}-T'_{1-\alpha})}) &= \big|E(e^{i\frac{\lambda}{\alpha}T_{(1-\alpha)}})\big|^2 \\
&= \exp\Big(-2|\lambda|^{1-\alpha}\alpha^\alpha\frac{1}{\alpha}\cos\Big(\frac{\pi}{2}(1-\alpha)\Big)\Big) \\
&= \exp\Big(-2|\lambda|^{1-\alpha}\alpha^\alpha\frac{\sin\left(\frac{\pi}{2}\alpha\right)}{\alpha}\Big) \\
&\underset{\alpha\to 0}{\longrightarrow} \exp\big(-\pi|\lambda|\big) = E(e^{i\lambda\pi C}).
\end{aligned}
$$

4 Proof of Theorem 1.4 (On the algebra of variables G, X, γ).

4.1 We first recall points 1), 2) and 3) of Theorem 1.4.

i) For every a, b, such that : $0 < a \leq b < 1$ there exists a r.v. $X_{a,b}$ such that :

$$E(e^{-\lambda X_{a,b}}) = \frac{b}{a} \frac{(1+\lambda)^a - 1}{(1+\lambda)^b - 1} \quad (\lambda \geq 0) \tag{4.1}$$

ii) For every $0 < a_1 < \cdots < a_n < 1$:

$$X_{a_1,a_n} \overset{\text{(law)}}{=} \sum_{i=1}^{n-1} X_{a_i,a_{i+1}} \tag{4.2}$$

where, on the RHS, the variables are assumed to be independent.

iii) The r.v.'s $X_{a,b}$ are infinitely divisible.

4.1.1 We prove (4.1).

4.1.1.a) For this purpose, we shall work in a slightly more general framework than what we strictly need.
We first recall that we use the term Bernstein function for a function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ of the form :

$$\Phi(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\nu(dx) \quad \text{for } \nu(dx) \geq 0 \text{ such that : } \int_0^\infty (1 \wedge x)\nu(dx) < \infty \tag{4.3}$$

In other terms, Φ is the Lévy exponent of a subordinator $(T_y,\ y \geq 0)$ with Lévy measure $\nu(dx)$, and without drift term, i.e. :

$$E(e^{-\lambda T_y}) = \exp\big(-y\Phi(\lambda)\big) \tag{4.4}$$

Lemma 4.1 : *Let Φ_1, Φ_2, Φ_3 denote three Bernstein functions which satisfy :*

i) $\Phi_1 = \Phi_3 \circ \Phi_2$

ii) $\displaystyle\int_0^\infty x\nu_3(dx) < \infty$*, where ν_3 denotes the Lévy measure associated with Φ_3.*

Then, there exists a positive r.v. X such that :

$$E(e^{-\lambda X}) = \frac{1}{C_3} \frac{\Phi_1(\lambda)}{\Phi_2(\lambda)}, \text{ with } C_3 = \int_0^\infty x\nu_3(dx)$$

Moreover :

$$E(e^{-\lambda X}) = \frac{1}{C_3} \frac{\Phi_1(\lambda)}{\Phi_2(\lambda)} = \frac{1}{C_3} E\Big(\int_0^\infty e^{-\lambda T_y^{(2)}} \overline{\nu}_3(y)dy\Big) \quad (\lambda \geq 0) \tag{4.5}$$

where, in (4.5),

$$(T_y^{(2)},\ y \geq 0) \text{ denotes the subordinator associated with } \Phi_2 \tag{4.6}$$

$$\bar{\nu}_3(y) = \nu_3\big([y,\infty)\big) \quad \textit{is the tail of } \nu_3 \tag{4.7}$$

Proof of Lemma 4.1 :
We have :

$$\begin{aligned}
\frac{\Phi_1(\lambda)}{\Phi_2(\lambda)} &= \frac{\Phi_3\big(\Phi_2(\lambda)\big)}{\Phi_2(\lambda)} = \int_0^\infty \Big(\frac{1-e^{-\Phi_2(\lambda)x}}{\Phi_2(\lambda)}\Big)\nu_3(dx) \\
&= \int_0^\infty \nu_3(dx) \int_0^x e^{-\Phi_2(\lambda)y} dy \\
&= \int_0^\infty e^{-\Phi_2(\lambda)y} \overline{\nu}_3(y)dy \qquad \text{(Fubini)}
\end{aligned}$$

Hence :

$$\frac{1}{C_3}\ \frac{\Phi_1(\lambda)}{\Phi_2(\lambda)} = \frac{1}{C_3}\ E\Big(\int_0^\infty e^{-\lambda T_y^{(2)}}\, \overline{\nu}_3(y)dy\Big) \quad \text{which proves Lemma 4.1}$$

once we have observed that :

$$\int_0^\infty \overline{\nu}_3(y)dy = \int_0^\infty dy \int_y^\infty \nu_3(dx) = \int_0^\infty x\nu_3(dx) = C_3.$$

4.1.1.b) We now prove (4.1) :
We denote for any $\delta \in]0,1[$:

$$\Phi_\delta(\lambda) = (1+\lambda)^\delta - 1 \tag{4.8}$$

Φ_δ is a Bernstein function since :

$$\Phi_\delta(\lambda) = \frac{\delta}{\Gamma(1-\delta)} \int_0^\infty (1-e^{-\lambda x})\ \frac{e^{-x}dx}{x^{\delta+1}} \tag{4.9}$$

(in fact, Φ_δ is the Lévy exponent of the Esscher transform $\big($cf [Sato]$\big)$ of the stable (δ) subordinator) with associated Lévy measure :

$$\nu_\delta(dx) = \frac{\delta}{\Gamma(1-\delta)}\ \frac{e^{-x}}{x^{\delta+1}}\ 1_{[0,\infty[}(x)dx \tag{4.10}$$

In particular :

$$\int_0^\infty x\nu_\delta(dx) = \delta \tag{4.11}$$

In the sequel, δ denotes either a, b, or $c := \dfrac{a}{b} < 1$, where : $0 < a < b < 1$. Note that :

$$\begin{aligned}
\Phi_c\big(\Phi_b(\lambda)\big) &= \big(\Phi_b(\lambda)+1\big)^c - 1 = \big((1+\lambda)^b - 1 + 1\big)^c - 1 \\
&= (1+\lambda)^{bc} - 1 = (1+\lambda)^a - 1 = \Phi_a(\lambda)
\end{aligned} \tag{4.12}$$

and that :

$$\int_0^\infty x\nu_c(dx) = c = \frac{a}{b} < \infty. \tag{4.13}$$

We may then use Lemma 4.1. with : $\Phi_1 = \Phi_a, \Phi_2 = \Phi_b$ and $\Phi_3 = \Phi_c$; given (4.12) and (4.13), we deduce the existence of an $\mathbb{R}_+$-valued r.v. such that :

$$E(e^{-\lambda X_{a,b}}) = \frac{1}{C_3} \frac{\Phi_1(\lambda)}{\Phi_2(\lambda)} = \frac{b}{a} \frac{(1+\lambda)^a - 1}{(1+\lambda)^b - 1} \quad (\lambda \geq 0).$$

4.1.1.c) We now prove (4.2) :
(4.2) follows immediately from the definition of $X_{a,b}$ and from the obvious formula :

$$\frac{a_n}{a_1} \frac{(1+\lambda)^{a_1} - 1}{(1+\lambda)^{a_n} - 1} = \prod_{i=1}^{n-1} \frac{a_{i+1}}{a_i} \frac{(1+\lambda)^{a_i} - 1}{(1+\lambda)^{a_{i+1}} - 1} \qquad (\lambda \geq 0) \tag{4.14}$$

4.1.1.d) We now prove the infinite divisibility of $X_{a,b}$:
We may write, from (4.2) :

$$X_{a,b} \overset{(\text{law})}{=} \sum_{i=0}^{n-1} X_{\left(a+i\frac{(b-a)}{n}, a+(i+1)\frac{(b-a)}{n}\right)} \tag{4.15}$$

We know $\big($cf [Loè], p.314-321$\big)$ that $X_{a,b}$ is infinitely divisible as soon as the following condition (called "uan") is satisfied :

$$\forall \varepsilon > 0, \quad \sup_{i=0,1,2,\cdots,n-1} P\big(X_{a+i\frac{(b-a)}{n},\, a+(i+1)\frac{(b-a)}{n}} > \varepsilon\big) \underset{n\to\infty}{\longrightarrow} 0 \tag{4.16}$$

But, by differentiation of (4.1), we obtain :

$$E(X_{a,b}) = \frac{b-a}{2} \tag{4.17}$$

Thus,

$$\delta_i^{(n)} := P(X_{a+\frac{i(b-a)}{n},\, a+\frac{(i+1)(b-a)}{n}} > \varepsilon) \leq \frac{b-a}{2n\varepsilon}$$

hence :

$$\sup_{i=0,1,2,\cdots,n-1} \delta_i^{(n)} \leq \frac{b-a}{2n\varepsilon} \underset{n\to\infty}{\longrightarrow} 0.$$

4.2 Remark 4.2.

4.2.1 (Self-decomposability of $X_{c,1}$; $0 < c < 1$).
Let $X_{c,1}$ denote a r.v. whose law is characterized by :

$$E(e^{-\lambda X_{c,1}}) = \frac{1}{c} \frac{(1+\lambda)^c - 1}{\lambda} \tag{4.18}$$

Then, $X_{c,1}$ is infinitely divisible, and its Lévy measure $\mu_{c,1}$ is given by :

$$\mu_{c,1}(dx) = (1-c)E(e^{-\frac{x}{G_c}}) \frac{dx}{x} \tag{4.19}$$

Proof of (4.19) : In order to prove that :

$$\frac{1}{c}\,\frac{(1+\lambda)^c-1}{\lambda}=\exp\Big(-\int_0^\infty(1-e^{-\lambda x})\mu_{c,1}(dx)\Big) \tag{4.20}$$

we take logarithmic derivatives of both sides ; thus :

$$\frac{1}{\lambda}-c\,\frac{(1+\lambda)^{c-1}}{(1+\lambda)^c-1}=\int_0^\infty e^{-\lambda x}x\,\mu_{c,1}(dx) \tag{4.21}$$

Denoting by (L) the LHS of (4.21), we get :

$$\begin{aligned}(L) &= \frac{1-c}{\lambda}+c\Big(\frac{1}{\lambda}-\frac{(1+\lambda)^{c-1}}{(1+\lambda)^c-1}\Big)\\ &= \frac{1-c}{\lambda}+\frac{c}{\lambda}\,\frac{(1+\lambda)^{c-1}-1}{(1+\lambda)^c-1}=(1-c)\Big[\frac{1}{\lambda}-\frac{1}{\lambda}\,\frac{c}{1-c}\,\frac{1-(1+\lambda)^{c-1}}{(1+\lambda)^c-1}\Big]\\ &= (1-c)\Big[\int_0^\infty e^{-\lambda x}dx-\Big(\int_0^\infty e^{-\lambda x}dx\Big)E(e^{-\lambda\mathfrak{e}G_c})\Big]\end{aligned}$$

from(1.19). We then deduce from (4.21) that :

$$\mu_{c,1}(dx)=(1-c)\frac{1}{x}\big[(\delta_0-\mu_{\mathfrak{e}G_c})*l_+\big]dx \tag{4.22}$$

where, in this expression, l_+ denotes Lebesgue measure on $\mathbb{R}_+$, and $\mu_{\mathfrak{e}G_c}$ the law of $\mathfrak{e}G_c$. The explicit computation of the convolution in (4.22) easily leads to (4.19). We note that the obtained formula :

$$E(e^{-\lambda X_{c,1}})=\exp\Big\{-(1-c)\int_0^\infty(1-e^{-\lambda x})E(e^{-\frac{x}{G_c}})\,\frac{dx}{x}\Big\} \tag{4.23}$$

may be compared with the "dual" formula (1.16) :

$$E(e^{-\lambda\Delta_c})=\exp\Big\{-(1-c)\int_0^\infty(1-e^{-\lambda x})E(e^{-xG_c})\,\frac{dx}{x}\Big\}$$

On the other hand, formula (4.23) implies that $X_{c,1}$ is self-decomposable.

4.2.2 (Self-decomposability of $X_{a,b}$ $(0<a<b<1)$).
Writing $X_{a,b}+X_{b,1}\overset{\text{(law)}}{=}X_{a,1}$ we deduce that the Lévy measure $\nu_{a,b}$ of $X_{a,b}$ equals :

$$\nu_{a,b}(dx)=\frac{1}{x}\big\{(1-a)E(e^{-\frac{x}{G_a}})-(1-b)E(e^{-\frac{x}{G_b}}\big\}\,dx \tag{4.24}$$

We prove now that $X_{a,b}$ is self-decomposable.
From (4.24), this assertion is equivalent to :

$$\varphi_{a,b}(x):=(1-a)E(e^{-\frac{x}{G_a}})-(1-b)E(e^{-\frac{x}{G_b}})$$

is a decreasing function (of x), or, by derivation :

$$(1-a)E\Big(\frac{1}{G_a}\,e^{-\frac{x}{G_a}}\Big)-(1-b)E\Big(\frac{1}{G_b}\,e^{-\frac{x}{G_b}}\Big)\geq 0$$

or, taking the Laplace transform in x of this expression :

$$\psi(\lambda) := (1-a)E\Big(\frac{1}{1+\lambda G_a}\Big) - (1-b)E\Big(\frac{1}{1+\lambda G_b}\Big)$$

is the Laplace transform of a positive function. But this assertion is an easy consequence of :
<u>Lemma 4.3</u> *For any* $0 < a < b < 1$ *and any* $u \in [0,1]$ *:*

$$(1-a)f_{G_a}(u) \geq (1-b)f_{G_b}(u) \tag{4.25}$$

Indeed, we have, with $\quad h(x) := (1-a)f_{G_a}(x) - (1-b)f_{G_b}(x)$:

$$\begin{aligned}\psi(\lambda) &= \int_0^1 \frac{dx}{1+\lambda x}\, h(x) = \int_0^1 h(x)dx \int_0^\infty \frac{1}{x} e^{-\lambda u - \frac{u}{x}} du \\ &= \int_0^\infty e^{-\lambda u} du \int_0^1 \frac{1}{x}\, e^{-\frac{u}{x}} h(x) dx\end{aligned}$$

<u>We now prove (4.25)</u>
By (1.17), we need to see that :

$$\begin{aligned}& a\ \sin(\pi a) \frac{(1-u)^{a-1}u^{a-1}}{(1-u)^{2a} - 2(1-u)^a u^a \cos(\pi\, a) + u^{2a}} \\ &= a\ \sin(\pi a) \frac{\big(\frac{1-u}{u}\big)^{a-1} u^2}{\big(\frac{1-u}{u}\big)^{2a} - 2\big(\frac{1-u}{u}\big)^a \cos(\pi\, a) + 1}\end{aligned}$$

is greater than the same expression where we replace a by b (with $a < b$).
Then, putting $\Big(\dfrac{1-u}{u}\Big) = x$, we have to prove :

$$\frac{a\sin(\pi a)}{b\sin(\pi b)} \geq \frac{x^{a+b} - 2x^b\cos(\pi\, a) + x^{b-a}}{x^{2b} - 2x^b \cos(\pi\, b) + 1} := \theta(x)$$

But it is easy to verify that $\theta(x) \underset{x\to+\infty}{\longrightarrow} 0,\ \theta(x) \underset{x\to 0}{\longrightarrow} 0$ and that $\theta(x)$ reaches its maximum for $x = 1$. The value of this maximum equals $\dfrac{1-\cos(\pi a)}{1-\cos(\pi b)}\cdot$ Hence, Lemma 4.2 will be proven if we show that :

$$\frac{a\sin(\pi a)}{b\sin(\pi b)} \geq \frac{1-\cos\pi(a)}{1-\cos\pi(b)} \qquad (0 < a < b < 1).$$

But, this relation is equivalent to :

$$\frac{1}{a}\, tg\Big(\frac{\pi a}{2}\Big) \leq \frac{1}{b}\, tg\Big(\frac{\pi b}{2}\Big)\ , \quad \text{i.e : the function } x \to \frac{1}{x}\, tg(x) \text{ is increasing on } \Big[0, \frac{\pi}{2}\Big[.$$

We have :

$$\begin{aligned}\frac{1}{a}\, tg\Big(\frac{\pi a}{2}\Big) &= \frac{1}{\pi} \sum_{n\geq 1} \frac{1}{(n-1/2)^2 - \frac{a^2}{4}} \\ &\leq \frac{1}{\pi} \sum_{n\geq 1} \frac{1}{(n-1/2)^2 - \frac{b^2}{4}} = \frac{1}{b}\, tg\Big(\frac{\pi b}{2}\Big)\end{aligned}$$

■

We note that we also have, for $0 < b < 1$ (we define $X_{0,b}$ as the limit in law of $X_{a,b}$, for $a \downarrow 0$) :

$$E(e^{-\lambda X_{0,b}}) = b\, \frac{\log(1+\lambda)}{(1+\lambda)^b - 1}$$

and $$E(e^{-\lambda X_{0,1}}) = \frac{\log(1+\lambda)}{\lambda}$$

From the latter relation, we easily deduce :

$$X_{0,1} \overset{\text{(law)}}{=} \mathfrak{e} \cdot U$$

with $\mathfrak{e}$ and U independent, $\mathfrak{e}$ standard exponential variable and U uniform on $[0,1]$. The density of $X_{0,1}$ equals :

$$f_{X_{0,1}}(x) = \Big(\int_x^\infty \frac{e^{-t}}{t}\, dt \Big)\, 1_{x \geq 0}$$

and its Lévy measure, from (1.29), is equal to :

$$\nu_{0,1}(dx) = \frac{1}{x}\, E\Big(\exp\big(-x(1+e^{\pi C})\big)\Big) dx$$

with C a standard Cauchy r.v., i.e. :

$$E(e^{-\lambda X_{0,1}}) = \frac{\log(1+\lambda)}{\lambda} = \exp\Big\{ -\int_0^\infty (1-e^{-\lambda x})\, \frac{E\Big(\exp -\big(x(1+e^{\pi C})\big)\Big)}{x}\, dx.$$

4.3 Remark 4.4. :

Let us come back to Lemma 4.1. Under the hypotheses of this Lemma, there exists an $\mathbb{R}_+$−valued r.v. X such that :

$$E(e^{-\lambda X}) = \frac{1}{C_3}\, \frac{\Phi_1(\lambda)}{\Phi_2(\lambda)} \tag{4.26}$$

It is natural to look for some criterion which ensures that X is infinitely divisible. Some further hypothesis on the Bernstein functions Φ_1, Φ_2, and Φ_3 is needed. Here is a framework which yields a positive answer to our question. For the sequel of the discussion in this remark, we refer the reader to Bertoin-Le Gall [B,LG]. Let us assume that the functions Φ_1 and Φ_2 are related to a continuous branching process. More precisely, let $\big(Z(t,x)\ ;\ t,x \geq 0\big)$ denote a continuous branching process, where t indicates the time parameter, and $x = Z(0,x)$ is the initial size of the population. Then :

$$E\Big[\exp\big(-\lambda Z(t,x)\big)\Big] = \exp\big(-x u(t,\lambda)\big) \tag{4.27}$$

where $u(t,\lambda)$ solves the differential equation :

$$\frac{\partial}{\partial t} u(t,\lambda) = -\psi\big(u(t,\lambda)\big) \tag{4.28}$$

with ψ denoting the branching mechanism of Z.

For each $t \geq 0$, $\lambda \to u(t,\lambda)$ is a Bernstein function and

$$u(t+s,\lambda) = u\big(t, u(s,\lambda)\big) \tag{4.29}$$

The relation (4.29) plays here the role of the relation $\Phi_1 = \Phi_3 \circ \Phi_2$, with :

$$\Phi_1(\lambda) = u(t+s,\lambda),\ \Phi_2(\lambda) = u(s,\lambda) \text{ and } \Phi_3(\lambda) = u(t,\lambda).$$

In this new set-up, we copy again the relation (4.14), which now writes :

$$u(t+s,\lambda) = \prod_{i=0}^{n-1} \frac{u\big(t+i\frac{s}{n},\lambda\big)}{u\big(t+(i+1)\frac{s}{n},\lambda\big)} \tag{4.30}$$

and we notice, as in point **4.1.1.d)** above, the infinite divisibility of the r.v. whose Laplace transform (in λ) equals :

$$\frac{u(t+s,\lambda)}{u(t,\lambda)}.$$

We also note that the Bernstein function $\Phi_a(\lambda) = (1+\lambda)^a - 1 \quad (0<a<1)$ coïncides with $u(t,\lambda)$, for $a = e^{-t}$, and ψ the branching mechanism :

$$\psi(q) = (1+q)\ \log(1+q)$$

$\big($see [BLG]$\big)$. This point 1) of Theorem 3 is a particular case of the situation that we just described in this Remark 4.4.

4.4 Remark 4.5. :

The relation (4.2) :

$$X_{a_1,a_n} \overset{\text{(law)}}{=} \sum_{i=1}^{n-1} X_{a_i,a_{i+1}} \quad (0 < a_1 < \cdots < a_n < 1) \tag{4.31}$$

where on the RHS the variables are independent invites to raise the following question : does there exist an homogeneous Markov process without positive jumps $(Z_t,\ t\geq 0)$ such that $X_{a,b}$ may be distributed as T_b under P_a, where P_a denotes the law of $(Z_t, t\geq 0)$, starting from a and $T_b = \inf\{t \geq 0 \ :\ Z_t > b\} \quad (a<b)$? The purpose of this Remark 4.5. is to show that such a process $(Z_t, t\geq 0)$ does not exist ; of course, it is also of interest to compare the present Remark 4.5. with the preceding one 4.4.

4.4.1 Proof of the non-existence of $(Z_t, t \geq 0)$:

Assume that such a process exists. Since :

$$E(e^{-\lambda X_{a,b}}) = \frac{b}{a}\, \frac{(1+\lambda)^a - 1}{(1+\lambda)^b - 1} \quad \lambda \geq 0, a<b \tag{4.32}$$

we would have :

$$E_a\Big[f(Z_{T_b}) \exp\Big(- \int_0^{T_b} \frac{\mathcal{L}f}{f}(Z_s)\Big) ds\Big] = f(a)$$

for any regular function f, i.e. :

$$E_a\Big(\exp - \int_0^{T_b} \frac{\mathcal{L}f}{f}(Z_s)ds\Big) = \frac{f(a)}{f(b)}$$

where P_a denotes the law of Z starting from $a, \mathcal{L}$ is the infinitesimal generator of Z, and f belongs to the (extended) domain of $\mathcal{L}$. Thus, we should have, for any $\lambda \geq 0$, that :

$$\Big(M_\lambda(t) := \frac{(1+\lambda)^{Z_t}}{Z_t} \exp -\lambda t, \quad t \geq 0\Big) \tag{4.33}$$

is a martingale. Hence :

$$E_a\Big(\frac{(1+\lambda)^{Z_t}-1}{Z_t}\Big) = \frac{(1+\lambda)^a - 1}{a}\, e^{\lambda t} \tag{4.34}$$

Writing : $l = \log(1+\lambda)$, i.e. : $\lambda = e^l - 1$, then (4.34) becomes :

$$\begin{aligned} E_a\Big(\frac{e^{lZ_t}-1}{Z_t}\Big) &= e^{(e^l-1)t}\Big(\frac{e^{al}-1}{a}\Big) &(4.35)\\ &= \frac{1}{a}\, E(e^{l(N_t+a)} - e^{lN_t}) &(4.36) \end{aligned}$$

where $(N_t, t \geq 0)$ denotes a standard Poisson process. Taking derivatives on both sides of (4.36) with respect to l, we obtain :

$$E_a(e^{lZ_t}) = \frac{1}{a}\, E\big[(N_t+a)e^{l(N_t+a)} - N_t e^{lN_t}\big] \ ;$$

hence, by Laplace inversion, the law of Z_t is identified as :

$$P_a(Z_t \in dx) = \frac{1}{a}\Big\{\sum_{n=0}^{\infty} \delta_{n+a}(dx)(n+a)e^{-t}\frac{t^n}{n!} - \sum_{n=0}^{\infty} \delta_n(dx) n\, e^{-t}\frac{t^n}{n!}\Big\} \tag{4.37}$$

But, the measure featured on the RHS of (4.37) is signed ; hence, (Z_t) does not exist.

4.4.2 Looking for signed measures on path space :

Denote, for $l \geq 0$: $\varphi_l(a) = \dfrac{\exp(la)-1}{a}$ $\quad (a > 0)$; then, define, for any $t \geq 0$:

$$P_t\varphi_l(a) = e^{(e^l-1)t}\varphi_l(a) \tag{4.38}$$

Our search of a process $(Z_t, t \geq 0)$ in **4.4.1.**, led us to the relation (4.35), which we now write as :

$$E_a\big[\varphi_l(Z_t)\big] = P_t\varphi_l(a) \tag{4.39}$$

On the other hand, the relation (4.38) leads to the semi-group property for $(P_t)_{t\geq 0}$, since :

$$\begin{aligned} P_s(P_t\varphi_l)(a) &= e^{(e^l-1)t}P_s(\varphi_l)(a) = e^{(e^l-1)(t+s)}\varphi_l(a) \\ &= P_{t+s}\varphi_l(a) &(4.40) \end{aligned}$$

Of course, from the relation (4.37), the semi-group (P_t) is not positive. Nonetheless, it is tempting to ask the question : does there exist a Markov "process" $\big(\Omega, (Z_t, t \geq 0), (P_a, a \geq 0)\big)$ with signed measures (P_a) on path space, such that the r.v.'s T_b, under P_a, are distributed as $X_{a,b}$?

4.5 Proofs of the points 3) and 4) of Theorem 1.4 :

i) For any $\alpha \in [0,1]$,

$$\mathfrak{e} \overset{\text{(law)}}{=} \mathfrak{e}_1 G_\alpha + \mathfrak{e}_2 G_{1-\alpha} \tag{4.41}$$

ii) For any $\alpha \in \left[\frac{1}{2}, 1\right]$ *:*

$$\mathfrak{e} G_\alpha \overset{\text{(law)}}{=} X_{1-\alpha,\alpha} + \gamma_{(1-\alpha)} \tag{4.42}$$

iii) For any $\alpha \in \left[0, \frac{1}{2}\right]$ *:*

$$X_{\alpha,1-\alpha} + \mathfrak{e} G_\alpha \overset{\text{(law)}}{=} \gamma_{(1-\alpha)} \tag{4.43}$$

As usual, it is understood that in these relations, whenever several r.v.'s are featured on one side, they are assumed independent. In the sequel of this work, this convention shall always be in force, without being stated each time. Moreover, $\mathfrak{e}$, with or without an index, indicates a standard exponential r.v. ; G_0 and G_1 denote the r.v.'s defined in Theorem1.2.

4.5.1 Proofs of (4.42) and (4.43) :

From (1.19), we get :

$$\begin{aligned} E(e^{-\lambda \mathfrak{e} G_\alpha}) &= E\Big(\frac{1}{1+\lambda G_\alpha}\Big) = \frac{\alpha}{1-\alpha}\,\frac{1-(1-\lambda)^{\alpha-1}}{(1+\lambda)^\alpha - 1} \quad (\lambda \geq 0) && (4.44)\\ &= \frac{\alpha}{1-\alpha}\,\frac{(1+\lambda)^{1-\alpha}-1}{(1+\lambda)^\alpha - 1}\cdot\frac{1}{(1+\lambda)^{1-\alpha}} && (4.45)\end{aligned}$$

· If $1-\alpha \leq \frac{1}{2}$, i.e. : $\alpha \geq \frac{1}{2}$, then (4.45) implies, from the definition (1.41) of the r.v.'s $X_{a,b}$:

$$E(e^{-\lambda \mathfrak{e} G_\alpha}) = E(e^{-\lambda X_{\alpha,1-\alpha}})\, E(e^{-\lambda \gamma_{(1-\alpha)}}), \quad \text{which yields (4.42).}$$

· If $\alpha \leq \frac{1}{2}$, (4.45) writes :

$$\begin{aligned} \frac{1}{(1+\lambda)^{1-\alpha}} &= E(e^{-\lambda \mathfrak{e} G_\alpha})\cdot\frac{1-\alpha}{\alpha}\,\frac{(1+\lambda)^\alpha - 1}{(1+\lambda)^{1-\alpha}-1}, \quad \text{hence :}\\ \frac{1}{(1+\lambda)^{1-\alpha}} &= E(e^{-\lambda \mathfrak{e} G_\alpha})\cdot E(e^{-\lambda X_{\alpha,1-\alpha}}), \quad \text{which yields (4.43).}\end{aligned}$$

We note that, if $\alpha > \frac{1}{2}$, (4.42) implies that $\mathfrak{e} G_\alpha$ is infinitely divisible.

4.5.2 Proof of (4.41) :
It is not difficult to show that (4.42) and (4.43) imply (4.41). However, we may also prove (4.41) directly, since :

$$E(e^{-\lambda\mathfrak{e}G_\alpha})\cdot E(e^{-\lambda\mathfrak{e}G_{1-\alpha}}) = E\Big(\frac{1}{1+\lambda G_\alpha}\Big)\cdot E\Big(\frac{1}{1+\lambda G_{1-\alpha}}\Big)$$
$$= \frac{\alpha}{1-\alpha}\cdot\frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-1}\cdot\frac{1-\alpha}{\alpha}\,\frac{1-(1+\lambda)^{-\alpha}}{(1+\lambda)^{\alpha-1}-1} = \frac{1}{1+\lambda} = E(e^{-\lambda\mathfrak{e}}).$$

5 Proof of Theorem 1.5. (The algebra of the r.v.'s $X_{a,b}, G_{\alpha,\beta}$ and gamma.)

We begin with the existence of the r.v.'s $G_{\alpha,\beta}$.

5.1 Proof of point 1) in Theorem 5 : *For any α,β, $0<\alpha,\beta<1$, there exists a r.v. $G_{\alpha,\beta}$ taking values in $[0,1]$, such that :*

i) $$E(e^{-\lambda\mathfrak{e}G_{\alpha,\beta}}) = E\Big(\frac{1}{1+\lambda G_{\alpha,\beta}}\Big) = \frac{\alpha}{1-\beta}\,\frac{1-(1+\lambda)^{\beta-1}}{(1+\lambda)^\alpha-1} \quad (\lambda\geq 0) \tag{5.1}$$

$$E\Big(\frac{1}{\lambda+G_{\alpha,\beta}}\Big) = \frac{\alpha}{1-\beta}\,\frac{\lambda^{\alpha-1}-(1+\lambda)^{\beta-1}\lambda^{\alpha-\beta}}{(1+\lambda)^\alpha-\lambda^\alpha} \quad (\lambda\geq 0) \tag{5.2}$$

ii) The density of $G_{\alpha,\beta}$ is

$$f_{G_{\alpha,\beta}}(u) = 1_{[0,1]}(u)\cdot\frac{\alpha}{\pi(1-\beta)}$$
$$\frac{(1-u)^\alpha u^{\alpha-1}\sin(\pi\alpha)+u^{2\alpha-\beta}(1-u)^{\beta-1}\sin(\pi\beta)+(1-u)^{\alpha+\beta-1}u^{\alpha-\beta}\sin\big(\pi(\alpha-\beta)\big)}{(1-u)^{2\alpha}-2(1-u)^\alpha u^\alpha\cos(\pi\alpha)+u^{2\alpha}} \tag{5.3}$$

5.1.1 Let us define :

$$F_{\alpha,\beta}(\lambda) = \frac{\alpha}{1-\beta}\,\frac{\lambda^{\alpha-1}-(1+\lambda)^{\beta-1}\lambda^{\alpha-\beta}}{(1+\lambda)^\alpha-\lambda^\alpha} \qquad (\lambda>0) \tag{5.4}$$

We shall show that $F_{\alpha,\beta}$ is the Stieltjes transform of the function $f_{G_{\alpha,\beta}}(u)$ defined by (5.3). To prove this, it suffices, with the help of the inverse Stieltjes transform, to show that :

$$\frac{F_{\alpha,\beta}(-u-i\eta)-F_{\alpha,\beta}(-u+i\eta)}{2i\pi} \underset{\eta\to 0_+}{\longrightarrow} f_{G_{\alpha,\beta}}(u) \quad (u\geq 0) \tag{5.5}$$

However, for $u\in[0,1]$:

$$\frac{1}{2i\pi}\big[F_{\alpha,\beta}(-u-i\eta)-F_{\alpha,\beta}(-u,+i\eta)\big]$$
$$= \frac{1}{2i\pi}\,\frac{\alpha}{1-\beta}\times\Bigg\{\frac{(-u-i\eta)^{\alpha-1}-(1-u-i\eta)^{\beta-1}(-u-i\eta)^{\alpha-\beta}}{(1-u-i\eta)^\alpha-(-u-i\eta)^\alpha}$$
$$-\frac{(-u+i\eta)^{\alpha-1}-(1-u+i\eta)^{\beta-1}(-u+i\eta)^{\alpha-\beta}}{(1-u+i\eta)^\alpha-(-u+i\eta)^\alpha}\Bigg\}$$

converges, as $\eta \downarrow 0$, to :

$$\frac{1}{2i\pi}\ \frac{\alpha}{1-\beta}$$

$$\left(\frac{-u^{\alpha-1}e^{-i\pi\alpha}-(1-u)^{\beta-1}u^{\alpha-\beta}e^{-i\pi(\alpha-\beta)}}{(1-u)^{\alpha}-u^{\alpha}e^{i\pi\alpha}}-\frac{-u^{\alpha-1}e^{i\pi\alpha}-(1-u)^{\beta-1}u^{\alpha-\beta}e^{i\pi(\alpha-\beta)}}{(1-u)^{\alpha}-u^{\alpha}e^{i\pi\alpha}}\right)$$

$$= \quad \frac{\alpha}{2i\pi(1-\beta} \cdot \frac{N}{(1-u)^{2\alpha}-2(1-u)^{\alpha}u^{\alpha}\cos(\pi\alpha)+u^{2\alpha}}, \quad \text{where } N \text{ is given by :}$$

$$\begin{aligned} N \ := \ & (-u^{\alpha-1}e^{-i\pi\alpha}-(1-u)^{\beta-1}u^{\alpha-\beta}e^{-i\pi(\alpha-\beta)})\,\big((1-u)^{\alpha}-u^{\alpha}e^{i\pi\alpha}\big) \\ & -\big(-u^{\alpha-1}e^{i\pi\alpha}-(1-u)^{\beta-1}u^{\alpha-\beta}e^{i\pi(\alpha-\beta)}\big)\,\Big((1-u)^{\alpha}-u^{\alpha}e^{-i\pi\alpha}\Big). \end{aligned}$$

Hence, for $u \in]0,1[$:

$$\frac{1}{2i\pi}\big[F_{\alpha,\beta}(-u-i\eta)-F_{\alpha,\beta}(-u+i\eta)\big] \underset{\eta\to 0_+}{\longrightarrow} \frac{\alpha}{\pi(1-\beta)}\cdot$$

$$\frac{u^{\alpha-1}(1-u)^{\alpha}\sin(\pi\alpha)+u^{2\alpha-\beta}(1-u)^{\beta-1}\sin(\pi\beta)+(1-u)^{\alpha+\beta-1}u^{\alpha-\beta}\sin\big(\pi(\alpha-\beta)\big)}{(1-u)^{2\alpha}-2(1-u)^{\alpha}u^{\alpha}\cos(\pi\alpha)+u^{2\alpha}}$$

$= f_{G_{\alpha,\beta}}(u)$, if $u \in]0,1[$, and it is not difficult to see that :

$$\frac{1}{2i\pi}\big[F_{\alpha,\beta}(-u-i\eta)-F_{\alpha,\beta}(-u+i\eta)\big] \underset{\eta\to 0_+}{\longrightarrow} 0\,, \quad \text{if } u>1.$$

5.1.2 We now prove that $f_{G_{\alpha,\beta}}$ is a probability density :
It is obvious that, for $\alpha \geq \beta$, $f_{G_{\alpha,\beta}}(u) \geq 0$, and it follows from elementary manipulation if $\alpha \leq \beta$.
Moreover, $\int_0^1 f_{G_{\alpha,\beta}}(u)\,du = 1$ since from (5.2) :

$$\begin{aligned} \int_0^1 f_{G_{\alpha,\beta}}(u)\,du &= \lim_{\lambda\to\infty} \frac{\alpha}{1-\beta}\ \lambda\ \frac{\lambda^{\alpha-1}-(1+\lambda)^{\beta-1}\lambda^{\alpha-\beta}}{(1+\lambda)^{\alpha}-\lambda^{\alpha}} \\ &= \lim_{\lambda\to\infty} \frac{\alpha}{1-\beta}\ \frac{1-(1+\frac{1}{\lambda})^{\beta-1}}{\big(1+\frac{1}{\lambda}\big)^{\alpha}-1} = 1. \end{aligned}$$

5.1.3 We now prove (5.1) :
It follows immediately from (5.2), since :

$$\begin{aligned} E\Big(\frac{1}{1+\lambda G_{\alpha,\beta}}\Big) &= \frac{1}{\lambda}\ E\Big(\frac{1}{\frac{1}{\lambda}+G_{\alpha,\beta}}\Big) = \frac{\alpha}{1-\beta}\ \frac{1}{\lambda}\,\frac{\big(\frac{1}{\lambda}\big)^{\alpha-1}-\big(\frac{1+\lambda}{\lambda}\big)^{\beta-1}\big(\frac{1}{\lambda}\big)^{\alpha-\beta}}{\big(\frac{1+\lambda}{\lambda}\big)^{\alpha}-\big(\frac{1}{\lambda}\big)^{\alpha}} \\ &= \frac{\alpha}{1-\beta}\ \frac{1-(1+\lambda)^{\beta-1}}{(1+\lambda)^{\alpha}-1} \end{aligned}$$

5.1.4 We prove that, for any $\alpha \in [0,1], G_{\alpha,\alpha} \overset{\text{(law)}}{=} G_{\alpha}$:
This follows immediately from the explicit value of the density $f_{G_{\alpha,\alpha}}$, as given by (5.3), or again from (5.1) :

$$E\Big[\frac{1}{1+\lambda G_{\alpha,\alpha}}\Big] = \frac{\alpha}{1-\alpha}\ \frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^{\alpha}-1} = E\Big(\frac{1}{1+\lambda G_{\alpha}}\Big).$$

5.1.5 We prove that, for $\alpha \in]0,1[, G_{\alpha,1-\alpha}$ is beta $(\alpha, 1-\alpha)$ distributed :
This follows immediately from the explicit value of the density $f_{G_{\alpha,1-\alpha}}$, or again from :

$$E\Big[\frac{1}{1+\lambda G_{\alpha,1-\alpha}}\Big] = \frac{1-(1+\lambda)^{-\alpha}}{(1+\lambda)^{\alpha}-1} = \frac{1}{(1+\lambda)^{\alpha}} = E\Big(\frac{1}{1+\lambda\beta_{(\alpha,1-\alpha)}}\Big)$$

5.2 Proof of point 2 in Theorem 1.5. Algebraic properties :

$$i) \qquad \textit{If } \alpha+\beta \geq 1, \textit{then } \mathfrak{e}G_{\alpha,\beta} \overset{\text{(law)}}{=} \gamma_{(1-\beta)} + X_{1-\beta,\alpha} \tag{5.6}$$

$$\textit{If } \alpha+\beta \leq 1, \textit{then } : \gamma_{1-\beta} \overset{\text{(law)}}{=} \mathfrak{e}G_{\alpha,\beta} + X_{\alpha,1-\beta} \tag{5.7}$$

ii) for any $0<\alpha,\beta,\gamma<1$,

$$\mathfrak{e}_1 G_{\alpha,\beta} + \mathfrak{e}_2 G_{\beta,\gamma} \overset{\text{(law)}}{=} \mathfrak{e}_1 G_{\alpha,\gamma} + \mathfrak{e}_2 G_{\beta} \tag{5.8}$$

$$iii) \textit{ If } \alpha+\beta \geq 1 \ : \ \gamma_{(1-\beta)} + X_{1-\beta,\alpha} + \mathfrak{e}_2 G_{\beta,\gamma} \overset{\text{(law)}}{=} \mathfrak{e}_1 G_{\alpha,\gamma} + \mathfrak{e}_2 G_{\beta} \tag{5.9}$$

$$\textit{If } \alpha+\beta \leq 1 \ : \ \gamma_{(1-\beta)} + \mathfrak{e}G_{\beta,\gamma} \overset{\text{(law)}}{=} \mathfrak{e}_1 G_{\alpha,\gamma} + \mathfrak{e}_2 G_{\beta} + X_{\alpha,1-\beta} \tag{5.10}$$

5.2.1 Proofs of (5.6) and (5.7) :
From the relation (5.1) :

$$E\Big(\frac{1}{1+\lambda G_{\alpha,\beta}}\Big) = \frac{\alpha}{1-\beta}\ \frac{1-(1+\lambda)^{\beta-1}}{(1+\lambda)^{\alpha}-1}\ ,$$

once both the numerator and denominator have been multiplied by $(1+\lambda)^{1-\beta}$, we obtain :

$$E(e^{-\lambda\mathfrak{e}G_{\alpha,\beta}}) = E\Big(\frac{1}{1+\lambda G_{\alpha,\beta}}\Big) = \Big(\frac{\alpha}{1-\beta}\ \frac{(1+\lambda)^{1-\beta}-1}{(1+\lambda)^{\alpha}-1}\Big)\cdot\frac{1}{(1+\lambda)^{1-\beta}} \tag{5.11}$$

If $\alpha+\beta \geq 1$, i.e. : $1-\beta \leq \alpha$, this relation writes :

$$E(e^{-\lambda\mathfrak{e}G_{\alpha,\beta}}) = E(e^{-\lambda X_{1-\beta,\alpha}})\ E(e^{-\lambda\gamma_{(1-\beta)}}), \quad \text{i.e. (5.6)}$$

If $\alpha+\beta \leq 1$, i.e. : $\alpha \leq 1-\beta$, we write (5.11) in the form :

$$\begin{aligned}\frac{1}{(1+\lambda)^{1-\beta}} &= E(e^{-\lambda\mathfrak{e}G_{\alpha,\beta}})\cdot\frac{1-\beta}{\alpha}\ \frac{(1+\lambda)^{\alpha}-1}{(1+\lambda)^{1-\beta}-1}, \quad \text{i.e.}\\ \frac{1}{(1+\lambda)^{1-\beta}} &= E(e^{-\lambda\mathfrak{e}G_{\alpha,\beta}})\cdot E(e^{-\lambda X_{\alpha,1-\beta}}),\end{aligned}$$

We have obtained (5.7).

5.2.2 Proofs of (5.8), (5.9) and (5.10) :
From (5.1), we write :

$$
\begin{aligned}
&E(e^{-\lambda\mathfrak{e}G_{\alpha,\beta}})\,E(e^{-\lambda\mathfrak{e}G_{\beta,\gamma}}) = E\Big(\frac{1}{1+\lambda G_{\alpha,\beta}}\Big)\cdot E\Big(\frac{1}{1+\lambda G_{\beta,\gamma}}\Big)\\
&= \frac{\alpha}{1-\beta}\,\frac{1-(1+\lambda)^{\beta-1}}{(1+\lambda)^{\alpha}-1}\cdot\frac{\beta}{1-\gamma}\,\frac{1-(1+\lambda)^{\gamma-1}}{(1+\lambda)^{\beta}-1}\\
&= \frac{\alpha}{1-\gamma}\,\frac{1-(1+\lambda)^{\gamma-1}}{(1+\lambda)^{\alpha}-1}\cdot\frac{\beta}{1-\beta}\,\frac{1-(1+\lambda)^{\beta-1}}{(1+\lambda)^{\beta}-1}\\
&= E(e^{-\lambda\mathfrak{e}G_{\alpha,\gamma}})\cdot E(e^{-\lambda\mathfrak{e}G_{\beta}}),\quad \text{i.e. (5.8)}
\end{aligned}
$$

Finally, the relations (5.9) and (5.10) follow easily from (5.8), (5.6) and (5.7).
The proof of point 2) *iv)* of Theorem 1.5 is obtained by similar arguments.

5.3 Remark 5.1. :

5.3.1 If we take $\gamma=\alpha$ in (5.8), we obtain :

$$\mathfrak{e}_1 G_{\alpha,\beta} + \mathfrak{e}_2 G_{\beta,\alpha} \overset{\text{(law)}}{=} \mathfrak{e}_1 G_{\alpha} + \mathfrak{e}_2 G_{\beta} \tag{5.12}$$

In particular, taking $\beta = 1-\alpha$ in (5.12), we obtain :

$$
\begin{aligned}
\mathfrak{e}_1 G_{\alpha} + \mathfrak{e}_2 G_{1-\alpha} &\overset{\text{(law)}}{=} \mathfrak{e}_1 G_{\alpha,1-\alpha} + \mathfrak{e}_2 G_{1-\alpha,\alpha}\\
&\overset{\text{(law)}}{=} \mathfrak{e}_1 \beta_{\alpha,1-\alpha} + \mathfrak{e}_2 \beta_{1-\alpha,\alpha}\\
&\overset{\text{(law)}}{=} \gamma_{\alpha} + \gamma_{(1-\alpha)} \overset{\text{(law)}}{=} \mathfrak{e}.
\end{aligned}
$$

This is our relation (4.41).
It is not difficult to show, after making some manipulations which are quite similar to the preceding ones, that (4.42) and (4.43) are particular cases of (5.9) and (5.10).

5.3.2 Of course, we did not find directly the explicit value of $f_{G_{\alpha,\beta}}$, as given by (5.3), with the help of the proof described in the above points **5.1.1** and **5.1.2.** Prior to that proof, we developed a heuristic computation which was quite similar to the one made in the above paragraph **2.2.3.**

6 The (δ, G) self-decomposable variables. Proofs of Theorems 1.6 and 1.7.

6.1 Let G be a positive r.v. such that :

$$E\Big[\log^{+}\Big(\frac{1}{G}\Big)\Big] < \infty \tag{6.1}$$

It is not difficult to show that (6.1) is equivalent to either of the following assertions :

- $$\int_1^{\infty}\frac{dx}{x}\,E\big[\exp(-xG)\big] < \infty \tag{6.2}$$

- $$\int_0^\infty (x \wedge 1) \frac{dx}{x} E(e^{-xG}) < \infty, \tag{6.3}$$

i.e. $\dfrac{dx}{x} E(e^{-xG}) 1_{x \geq 0}$ is the Lévy measure of a subordinator.

- $$\Big(\log \Big(1 + \frac{\lambda}{G} \Big) \Big) < \infty \quad \text{for one (hence any) value of } \lambda > 0 \tag{6.4}$$

We may then formulate, thanks to the Lévy-Khintchine formula, the following :

Definition 6.1. : Let $\delta > 0$, and G an $\mathbb{R}_+$-valued r.v. which satisfies (6.1). We shall say that a $\mathbb{R}_+$-valued r.v. Δ is (δ, G) self-decomposable if, for every $\lambda \geq 0$:

$$E(e^{-\lambda \Delta}) = \exp \Big\{ -\delta \int_0^\infty (1 - e^{-\lambda x}) \, E(e^{-xG}) \frac{dx}{x} \Big\} \tag{6.5}$$

The equality (6.5) may also be written as :

$$E(e^{-\lambda \Delta}) = \exp \Big\{ -\delta E \Big(\log \Big(1 + \frac{\lambda}{G} \Big) \Big) \Big\} \tag{6.6}$$

the latter formula (6.6) being obtained, e.g., as an application of the Frullani integral (see [L], p. 6). In fact, we thought of Definition 6.1 after considering formula (1.16), which, in our terminology may be stated as : the r.v. Δ_α is $(1-\alpha, G_\alpha)$ self-decomposable.

6.2 The notion of (δ, G) self-decomposability is related quite naturally with the standard gamma subordinator.

Statement and Proof of Theorem 1.6 : (A link between the standard gamma subordinator and the (δ, G) self-decomposability).

Let $(\gamma_t, t \geq 0)$ denote the standard gamma subordinator, whose Lévy-Khintchine representation writes :

$$E(e^{-\lambda \gamma_t}) = \frac{1}{(1+\lambda)^t} = \exp \big(-t \log(1+\lambda) \big) \quad (\lambda, t \geq 0) \tag{6.7}$$

and let $h : [0, \infty[\longrightarrow [0, \infty[$ Borel.

1) Define :

$$\Delta_h := \int_0^\infty h(u) d\gamma_u \tag{6.8}$$

Then, Δ_h is a.s. finite if and only if :

$$\int_0^\infty \log \big(1 + h(u) \big) du < \infty \tag{6.9}$$

2) Assuming that the hypothesis (6.9) is satisfied, then Δ_h is self-decomposable, with Lévy-Khintchine representation :

$$E(e^{-\lambda\Delta_h}) = \exp\Big(- \int_0^\infty (1-e^{-\lambda x})F_h(x)\,\frac{dx}{x}\Big) \tag{6.10}$$

with

$$F_h(x) = \int_0^\infty \exp\big(- \frac{x}{h(u)}\big)du \tag{6.11}$$

3) For any r.v. $G > 0$ satisfying (6.1), there exists h satisfying (6.9) such that :

$$\delta E(e^{-xG}) = F_h(x) = \int_0^\infty e^{-\frac{x}{h(u)}}du$$

In other terms, every (δ, G) self-decomposable r.v. may be written in the form (6.8), for a well-chosen function h.

Recall (cf. the remark following the statement of Theorem 1.6 in part 1.8 of the Introduction) that :

- the function h, whose existence is asserted in the above point 3) is explicitly given in terms of δ and G via the formula :

$$h(u) = \frac{1}{\mathcal{G}^{-1}\left(\frac{u}{\delta}\right)}\,,\ \text{for } u\in(0,\delta)\,,\ \text{and } 0, \text{ for } u>\delta\,;$$

- the Laplace transform ψ_{Δ_h} of the r.v. $\Delta_h$is hyperbolically completely monotone.

6.2.1 We prove (6.9) and (6.10) :
By a density argument, it suffices to consider h continuous,with compact support. Then, one has :

$$\begin{aligned}
E(e^{-\lambda\int_0^\infty h(u)d\gamma_u}) &= \lim E(e^{-\lambda\sum h(t_i)(\gamma_{t_{i+1}}-\gamma_{t_i}}) \\
&= \lim\,\exp\Big\{-\sum(t_{i+1}-t_i)\log\big(1+\lambda h(t_i)\big)\Big\} \qquad \text{from (6.7)} \\
&= \exp\Big(-\int_0^\infty \log\big(1+\lambda h(t)dt\big) \qquad\qquad (6.12)\\
&= \exp\Big(-\int_0^\infty dt\int_0^\infty e^{-x}\frac{dx}{x}(1-e^{-\lambda h(t)x}\big)
\end{aligned}$$

since, for every $v \geq 0$, the Frullani integral $\big($cf [Leb] p. 6$\big)$ gives :

$$\log(1+v) = \int_0^\infty \frac{dx}{x}e^{-x}(1-e^{-vx})dx$$

Hence, making the change of variables $h(t)\,x = y$, and then applying Fubini's Theorem :

$$\begin{aligned} E\Big(\exp\big(-\lambda\int_0^\infty h(u)d\gamma_u\big)\Big) &= \exp\Big(-\int_0^\infty dt\int_0^\infty e^{-x}\frac{dx}{x}(1-e^{-\lambda h(t)x})\Big) \\ &= \exp\Big(-\int_0^\infty dt\int_0^\infty e^{-\frac{y}{h(t)}}\frac{dy}{y}(1-e^{-\lambda y})\Big) \\ &= \exp\Big(-\int_0^\infty (1-e^{-\lambda y})\frac{dy}{y}\Big(\int_0^\infty e^{-\frac{y}{h(t)}}dt\Big)\Big) \end{aligned}$$

which proves both (6.9) and (6.10).

6.2.2 <u>We prove point 3) of Theorem 1.6</u> :
Assume now that G satisfies (6.1), or (6.4), and $\delta > 0$. Let us consider the probability space obtained from the unit interval $[0,1]$, fitted with Lebesgue's measure, and realize G in the form :

$$G(w) = \left(\frac{1}{h}\right)(\delta w), \quad w \in [0,1]. \tag{6.13}$$

for a well-chosen function h, with support in $[0,\delta]$. Then we obtain :

$$\delta\; E(e^{-xG}) = \delta\int_0^1 e^{-\frac{x}{h(\delta u)}}\,du = \int_0^\delta e^{-\frac{x}{h(v)}}\,dv = \int_0^\infty e^{-\frac{x}{h(v)}}\,dv \tag{6.14}$$

Thus :

$$\begin{aligned} E(e^{-\lambda\int_0^\infty h(u)d\gamma_u}) &= \exp\Big\{-\int_0^\infty(1-e^{-\lambda y})\frac{dy}{y}\Big(\int_0^\infty e^{-\frac{y}{h(v)}}\,dv\Big)\Big\} \\ &= \exp\Big\{-\delta\int_0^\infty(1-e^{-\lambda y})\frac{dy}{y}E(e^{-xG})\Big\} \end{aligned}$$

Finally, it is clear, as a consequence of the definition (6.13) that :

$$E\Big(\log^+\frac{1}{G}\Big) < \infty \Longleftrightarrow E\Big(\log\Big(1+\frac{1}{G}\Big)\Big) < \infty \Longleftrightarrow \int_0^\infty \log\big(1+h(u)\big)du < \infty.$$

6.2.3 <u>Proof of Theorem 1.7</u> :
Mutatis mutandis, it is exactly the same as the proof of point 3 of Theorem 1 (cf, Proposition 2.3 and paragraph **2.3.3** above).

6.2.4 <u>Proof of Theorem 1.8</u> :
<u>Definition 6.3</u> : A function $F :]0,\infty[\longrightarrow \mathbb{R}_+$, which belongs to C^1, satisfies (ST,δ) if :

i) F admits an holomorphic extension to $\mathbb{C}\backslash]-\infty,0]$ (6.15)

ii) For every $u > 0$:

$$\begin{aligned} &\lim_{\eta\to 0_+} F(-u+i\eta) := F_+(u) \ \underline{\text{exists and in continuous}} \\ &\text{resp.} : \lim_{\eta\to 0_+} F(-u-i\eta) := F_-(u), \ \underline{\text{exists and in continuous}} \\ &\mathrm{Im}\big(F_-(u)-F_+(u)\big) \geq 0 \quad \text{for every } u > 0. \end{aligned} \tag{6.16}$$

$$iii) \quad \lim_{\substack{\lambda\to\infty \\ \lambda \text{ real}}} \lambda F(\lambda) = \delta > 0 \tag{6.17}$$

Let Δ denote a positive r.v. with Laplace transform ψ :

$$E(e^{-\lambda\Delta}) = \psi(\lambda), \quad \lambda \geq 0.$$

We assume that $F := \dfrac{\psi'}{\psi}$ satisfies (ST, δ).

6.2.4.a) <u>We show that :</u> $f(u) := \dfrac{1}{2\pi\delta}\ \big(\mathrm{Im}\big(F_-(u) - F_+(u)\big)$ <u>defines a probability density on $\mathbb{R}_+$ and that Δ is (δ, G) self-decomposable, where G is a r.v. with density f</u> :
In fact, we have already made this proof when we showed the existence of the r.v.'s G_α (paragraph **2.2.1**) and of the r.v.'s $G_{\alpha,\beta}$ (paragraph **5.1**). We now summarize the important points of this proof :

- By inversion of the Stieltjes transform, we have :

$$Sf(\lambda) = \int_0^\infty \frac{f(u)du}{\lambda + u} = -\frac{1}{\delta}\ \frac{\psi'}{\psi}(\lambda)$$

- f is positive $\big($from (6.16)$\big)$ and has integral 1 $\big($from (6.17)$\big)$.

- Let G denote a r.v. with density f. Then :

$$\delta E\Big(\frac{1}{\lambda + G}\Big) = -\frac{\psi'}{\psi}(\lambda), \quad \text{hence :}$$

$$-\delta \int_0^\infty e^{-\lambda x} E(e^{-xG}) dx = -\frac{\psi'}{\psi}(\lambda) \ ; \quad \text{consequently, by integration :}$$

$$E(e^{-\lambda\Delta}) = \psi(\lambda) = \exp\Big\{ -\delta \int_0^\infty (1 - e^{-\lambda x}) E(e^{-xG})\ \frac{dx}{x}\ \Big\}$$

r.v.	density (*only the density is given, not the LT.)	Laplace Transform	Stieltjes Transform	Lévy measure
$\Delta_\alpha \quad (0<\alpha<1)$	$\dfrac{\alpha}{\Gamma(1-\alpha)}\, x^{-(\alpha+1)}(1-e^{-x})1_{x\geq 0}$	$(1+\lambda)^\alpha-\lambda^\alpha$		$(1-\alpha)\dfrac{E(e^{-xG_\alpha})}{x}dx$
$G_\alpha \quad (0<\alpha<1)$	* $\dfrac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi}\,\dfrac{x^{\alpha-1}(1-x)^{\alpha-1}}{(1-x)^{2\alpha}-2(1-x)^\alpha\, x^\alpha\cos(\pi\alpha)+x^{2\alpha}}\;1_{[0,1]}(x)$		$\dfrac{\alpha}{1-\alpha}\;\dfrac{\lambda^{\alpha-1}-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-\lambda^\alpha}$	non inf. div.
$G_{1/2}$	$\dfrac{1}{\pi}\,\dfrac{1}{\sqrt{x(1-x)}}\;1_{[0,1]}(x)$		$\frac{1}{\sqrt{\lambda(1+\lambda)}}$	non inf. div.
$G_{1/p} \quad p\in\mathbb{N}$ $p\geq 2$	* $\dfrac{1}{\pi(p-1)}\displaystyle\sum_{k=1}^{p-1}\sin\left(\frac{\pi k}{p}\right)x^{\frac{k}{p}-1}(1-x)^{-\frac{k}{p}}\;1_{[0,1]}(x)$			non inf. div.
G_1	$1_{[0,1]}(x)$	$\dfrac{1}{\lambda}(1-e^{-\lambda})$	$\log\,(1+\lambda)$	non inf. div.
$G_0 \overset{(\text{law})}{=} \dfrac{1}{1+\exp\pi C}$ C standard Cauchy	* $\dfrac{1}{x(1-x)}\;\dfrac{1}{\pi^2+\left(\log\frac{1-x}{x}\right)^2}\;1_{[0,1]}(x)$		$\dfrac{1}{\lambda(1+\lambda)}\;\dfrac{1}{\log\left(\frac{1+\lambda}{\lambda}\right)}$	non inf. div.
$G_{\alpha,\beta} \quad (0<\alpha,\beta<1)$	* $\frac{\alpha}{\pi(1-\beta)}\;\frac{(1-x)x^{\alpha-1}\sin(\pi\alpha)+x^{2\alpha-\beta}(1-x)^{\beta-1}\sin(\pi\beta)+(1-x)^{\alpha+\beta-1}x^{\alpha-\beta}\sin\left(\pi(\alpha-\beta)\right)}{(1-x)^{2\alpha}-2(1-x)^\alpha x^\alpha\cos(\pi\alpha)+x^{2\alpha}}$ $\cdot\, 1_{[0,1]}(x)$		$\dfrac{\alpha}{1-\beta}\;\dfrac{\lambda^{\alpha-1}-(1+\lambda)^{\beta-1}\lambda^{\alpha-\beta}}{(1+\lambda)^\alpha-\lambda^\alpha}$	non inf. div.

r.v.	density (*only the density is given, not the LT.)	Laplace Transform	Stieltjes Transform	Lévy measure
$G_{\alpha,\alpha} \overset{(\text{law})}{=} G_\alpha$			$\dfrac{\alpha}{1-\alpha} \; \dfrac{\lambda^{\alpha-1} - (1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha - \lambda^\alpha}$	
$G_{\alpha,1-\alpha} \overset{(\text{law})}{=} \beta_{\alpha,1-\alpha}$	$\frac{\sin(\pi\alpha)}{\pi} x^{\alpha-1}(1-x)^{-\alpha}\, 1_{[0,1]}(x)$		$\frac{\lambda^{\alpha-1}}{(1+\lambda)^\alpha}$	non inf. div.
$X_{a,b}$ $(0 < a \leq b < 1)$		$\frac{a}{b} \, \frac{(1+\lambda)^a - 1}{(1+\lambda)^b - 1}$		$\frac{1}{x}\left[(1-a)\, E\big(e^{-\frac{x}{G_a}}\big) - (1-b)\, E\big(e^{-\frac{x}{G_b}}\big)\right] dx$
$X_{a,1}$ $(0 < a < 1)$		$\frac{1}{a} \, \frac{(1+\lambda)^a - 1}{\lambda}$		$(1-a)\, \frac{E\left(e^{-\frac{x}{G_a}}\right)}{x}\, dx$
$X_{0,1} \overset{(\text{law})}{=} \mathfrak{e} \cdot U$	$\left(\int_x^\infty \frac{e^{-t}}{t}\, dt\right) 1_{x \geq 0}$	$\frac{\log(1+\lambda)}{\lambda}$		$\frac{1}{x}\, E(\exp -x(1+e^{\pi C}))\, dx$ C standard Cauchy
$X_{0,b}$ $(0 < b < 1)$		$\frac{\log(1+\lambda)}{(1+\lambda)^b - 1}$		$\frac{1}{x}\left\{E\left[\exp\left(-x\left(1+e^{\pi C}\right)\right)\right] - (1-b)\, E\big(e^{-\frac{x}{G_b}}\big)\right\} dx$

Main properties

$\Delta_\alpha \overset{(\text{law})}{=} \int_0^\infty e^{-t}\, dY_t$, $(Y_t, t \geq 0)$ is an (α, K_α) compound Poisson process, with $K_\alpha \overset{(\text{law})}{=} \frac{\mathfrak{e}}{G_\alpha}$

$\Delta_\alpha \overset{(\text{law})}{=} U^{1/\alpha}(\Delta_\alpha + K_\alpha)$ $(U, \Delta_\alpha, K_\alpha$ independent, and U uniform on $[0,1])$

$\mathfrak{e} \overset{(\text{law})}{=} \mathfrak{e}_1 G_\alpha + \mathfrak{e}_2 G_{1-\alpha}$; if $\alpha \in \left[\frac{1}{2}, 1\right]$ $\mathfrak{e} G_\alpha \overset{(\text{law})}{=} \gamma_{1-\alpha} + X_{1-\alpha,\alpha}$; if $\alpha \in \left[0, \frac{1}{2}\right]$, $X_{\alpha,1-\alpha} + \mathfrak{e} G_\alpha \overset{(\text{law})}{=} \gamma_{1-\alpha}$

$\mathfrak{e}_1 G_{\alpha,\beta} + \mathfrak{e}_2 G_{\beta,\gamma} \overset{(\text{law})}{=} \mathfrak{e}_1 G_{\alpha,\gamma} + \mathfrak{e}_2 G_\beta$

if $\alpha + \beta \geq 1$, $\mathfrak{e} G_{\alpha,\beta} \overset{(\text{law})}{=} \gamma_{1-\alpha} + X_{1-\beta,\alpha}$; if $\alpha + \beta \leq 1$, $\gamma_{1-\beta} \overset{(\text{law})}{=} \mathfrak{e} G_{\alpha,\beta} + X_{\alpha,1-\beta}$

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